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ON THE ROLE OF AN INTERSECTION PROPERTY IN MEASURE THEORY — II.*

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4. Uniqueness and equivalence of measures in groups. Although the phenomenon of the uniqueness of measures in groups with topological structure has been the object of many publications [2], it has been only studied for invariant measures in locally compact topological groups. In this section stronger results are derived for a wider class of measures from the general theorems proved in section **3**.

To this end the basic set G is assumed to be an abstract group and the relation R is specialized to mean the left congruence of subsets of G (i. e., any relation of the form $B=gA$ with $g \in G$). Then the terms *left 0-invariant measure, left variation* and *left intersection property* are defined to mean a measure 0-invariant under R, the variation of a measure under R and the intersection property modulo R, respectively. Symmetric definitions are used if R is the right congruence of subsets of G.

Let $\overline{P}$ denote the σ-field *generated* by a family P of subsets of G, i. e. the least σ-field of subsets of G that contains P. A class K of measures defined on $\overline{P}$ is said to have the *Equivalence Porperty* if every measure in K is both left and right 0-invariant and if all elements of K have the same null sets. The class K is said to have the *Left* or *Right Uniqueness Property* if a measure in K is uniquely determined by its left or right variation, respectively. A class of measures which has the equivalence property as well as both the left and right uniqueness property is termed *harmonic*.

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(2) Cf. [1] - [6]. Numbers in brackets refer to bibliography at the end.

The goal of this section is twofold: 1. to prove that the family of all either left or right 0-invariant measures defined on the domain of Haar's measure is harmonic (theorem 9) and thus to strengthen the already classical theorem on the uniqueness of Haar's measure; 2. to establish the same property for other families of measures in groups (theorems 8, 10 and 11).

We begin with a theorem which strengthens a result of A. Weil ([6], pp. 148-149). Its formulation requires the following definitions, in which $P \times P$ denotes the family of the combinatorial products $A \times B$ formed for all pairs A, B of elements of a family of sets P.

DEFINITION 1. Let S be a σ-field of subsets of an abstract group G. A family P of elements of S is called *left-regular relative* to S if it is left invariant (i. e., if $A \in P$, $g \in G$ implies $gA \in P$) and if every pair A, B of elements of P satisfies the following conditions: (*i*) there is a set $\bar{A}$ in $\bar{P}$ which contains A^{-1}; (*ii*) the transformation $T(x, y) = (yx, y)$ of $G \times G$ onto itself sends $A \times B$ into an element of the σ-field $\overline{S \times S}$. — A *right regular* family is defined in a symmetric way.

DEFINITION 2. A measure m defined on a σ-field S' is called the *contraction to* S' of a measure n defined on a σ-field S containing S' if $X \in S'$ implies $m(X) = n(X)$.

DEFINITION 3. A measure m is called *σ-finite* if every m-measurable set is the union of a sequence of sets of finite m-measure.

DEFINITION 4. If K is a class of measures defined on a σ-field S of subsets of an abstract group G, then a class L of measures which are obtained by extending the elements of K to any fixed left and right invariant σ-field containing S and contained in the completion of S with respect to some element of K is called a *perfection* of K provided that L is closed under addition.

THEOREM 7. Let S be a σ-field of subsets of an abstract group G. If a family P of elements of S is both left and right regular relative to S and if it either is invariant under the transformation $x \to x^{-1}$ or generates S, then the class K of all left or right 0-invariant measures which are contractions to $\bar{P}$ of σ-finite measures defined on S is harmonic as well as all of its perfections.

PROOF: Under the above assumptions the σ-field $\bar{P}$ is left regular relative to S.

In fact, if P_1 and P_2 are families of subsets of G, then any one-to-one transformation t of G onto G which maps P_1 into P_2 also maps $\overline{P}_1$ into $\overline{P}_2$ (for $t^{-1}(\overline{P}_2)$ is a σ-field containing P_1).

In particular, since t can be any transformation $x \to gx$ with g in G, $\overline{P}$ is left invariant, and since G can be replaced by $G \times G$ and t by T, T maps $\overline{P \times P}$ into $\overline{S \times S}$ which implies that $\overline{P}$ satisfies the left regularity condition (*ii*) (since the identity $\overline{P \times P} = \overline{\overline{P} \times \overline{P}}$ is easily verified). Finally, every element A of $\overline{P}$ being contained in the union of a sequence $\{A_i\}$ of elements of P, A^{-1} is contained in the union $\tilde{A}$ of the elements $\tilde{A}_i \supset A_i^{-1}$ of $\overline{P}$ hence $\overline{P}$ satisfies the condition (*i*).

Now let m be a left 0-invariant element of K which is the contraction of a σ-finite measure n defined on S, and let A and B be elements of $\overline{P}$ of positive m-measure.

The set $E = T(A \times B)$ being in $\overline{S \times S}$, the sets $E_x = \{y : (x, y) \in E\}$, $E^y = \{x : (x, y) \in E\}$ and the functions $n(E_x)$ and $n(E^y)$ are S-measurable for all x, y in G, and we have

$$\int n(E_x)\, dn = \int n(E^y)\, dn. \tag{4}$$

(Cf. [1, pp. 141-143]). An easy computation shows that:

1. $E_x = xA^{-1} \cap B$ for all x in G. Therefore the set A^{-1} equals E_e for $B = \tilde{A} \supset A^{-1}$ (where e is the unity of G) and thus is in S; in view of our assumptions it even is in $\overline{P}$;

2. E^y equals yA for all y in B and is void for all y in $G - B$. Therefore (4) yields

$$\int m(xA^{-1} \cap B)\, dm = \int_B m(yA)\, dm. \tag{5}$$

The right side of (5) being positive, so is its left side. Hence whenever A is of positive m-measure, so is both A^{-1} and $(x^{-1}A^{-1})^{-1} = Ax$ for every x in G (by left 0-invariance of m). This implies that the measure m is right 0-invariant.

Moreover, the substitution of A^{-1} for A in (5) shows that $\int m(xA \cap B)\, dm$ is positive. Therefore the integrand of this integral does not vanish identically, which implies the left intersection property of m.

From these properties of m immediately follow the corresponding properties of the extension of m to any left and right 0-invariant σ-field containing $\overline{P}$ and contained in the completion of $\overline{P}$ with respect to m. By symmetry, «left» and «right» are interchangeable in the formulation of these properties, and theorems 2 and 3 complete the proof.

To apply theorem 7 to general topological groups (3) we say that a subset F of a topological space G has the *Lindelöf property in* G if in every family of open sets whose union contains F there is a countable subfamily whose union also contains F.

LEMMA 1. A family P of G_δ-sets contained in a σ-field S of subsets of a general topological group G satisfies the left regularity condition (*ii*) relative to S if (*a*) to arbitrary two elements A and B of P there is a subgroup $G^* \supset A \cup B$ such that $A \times B$ has the Lindelöf property in the topological product $G^* \times G^*$ (with the topology of G^* induced by G) and (*b*) the family $S^* = S \cap G^*$ of all sets of the form $X \cap G^*$ with X in S is contained in S and contains a complete system of neighborhoods of every point in G^*.

PROOF: The following facts are easily verified:

1. Both G^* and $G^* \times G^*$ are general topological groups.
2. The combinatorial product $A \times B$ is a G_δ-set in $G^* \times G^*$.
3. The σ-field $\overline{S^* \times S^*}$ contains a complete system of neighborhoods of every point in $G^* \times G^*$.
4. The image E of $A \times B$ under the transformation $T(x, y) = (yx, y)$ of $G^* \times G^*$ onto $G^* \times G^*$ has the Lindelöf property in $G^* \times G^*$ and is the intersection of a sequence $\{0_n\}$ of open subsets of $G^* \times G^*$ (these properties being invariant under homeomorphisms).

The last two facts imply that E is contained in the union E_n of a sequence of elements of $\overline{S^* \times S^*}$ which are subsets of 0_n $(n = 1, 2, \cdots)$. Therefore E is the intersection of the sets E_n hence belongs to $\overline{S^* \times S^*} \subset \overline{S \times S}$, which completes the proof.

THEOREM 8. Let P and Q be the families of all compact G_δ-sets and all compact sets of an arbitrary topological group G, respectively. Then the class K of all left or right 0-invariant measures on $\overline{P}$ which are contractions of σ-finite measures defined on $\overline{Q} = S$ is harmonic as well as all of its perfections.

PROOF: The family P is clearly both left invariant and invariant under the transformation $x \to x^{-1}$. To prove that it is left regular relative to S one verifies, for arbitrary two of its elements A and B, that: 1. the set $G^* = \bigcup_{n=1}^{\infty} D^n$ with $D = (A \cup B) \cup (A \cup B)^{-1}$ is a subgroup

(3) i. e., groups G which are topological spaces such that $y^{-1}x$ is a continuous mapping of the topological product $G \times G$ into G. The term «topological group» is reserved for general topological groups which are Hausdorff spaces.

of G which belongs to S and contains $A \cup B$; this implies that the family $S^* = S \cap G^*$ is contained in S; 2. if the topology of G^* is induced by G, then the sets A and B are compact in G^* hence the set $A \times B$ has the Lindelöf property in the topological product $G^* \times G^*$. Moreover, each of the sets D^n being the union of a finite number of closed compact sets, G^* is the union of a sequence of such sets, hence every set closed in G^* belongs to the family S^*. Being a topological group, G^* has a complete system of closed neighborhoods of every of its points (cf. [1, p. 10]). Since this system is contained in S^*, lemma 1 implies that the family P is left regular relative to S, and theorem 7 together with symmetry reasons completes the proof.

Corollary. Under the assumptions of theorem 8 let G have the following *Property of Kakutani-Kodaira: The σ-field* $\bar{Q}$ *is contained in the completion of the σ-field* $\bar{P}$ *with respect to some element of* K. Then the class L of all σ-finite left or right 0-invariant measures defined on $\bar{Q}$ is harmonic.

For L clearly is a perfection of K.

Since every locally compact topological group has the property of Kakutani-Kodaira (cf. [3], [1] and [5]), the following theorem holds.

Theorem 9. If Q is the family of all compact subsets of an arbitrary locally compact topological group, then the class L of all σ-finite left or right 0-invariant measures defined on $\bar{Q}$ is harmonic.

Obviously the uniqueness of Haar's measure (i. e., of any element of L with the left variation 1) is a special case of this theorem. (Observe that this uniqueness follows without the usual assumption that the measure is finite on compact sets).

Another application of theorem 7 is based on the following lemma.

Lemma 2 (4). If the family Q of all closed compact subsets of a topological space G contains a complete system of neighborhoods of every point in G, then so does the family P of all closed compact G_δ-sets of G.

In fact, since to every A in Q and to every open set $B \supset A$, there is a set Z in Q with $A \subset Z^0 \subset Z \subset B$ (where Z^0 is the interior of Z), there also is an open set $0 = (A, B)$ with compact closure such that we have $A \subset 0 \subset \bar{0} \subset B$ (for instance, Z^0 is such a set). Let $0_0 = B$ and $0_{n+1} = (Z, 0_n)$ for $n = 0, 1, 2, \cdots$. Then the set $D = \bigcap_1^\infty 0_n$ is a G_δ and

(4) For the special case that G is a locally compact Hausdorff space, this lemma is proved in [1, pp. 217-8] in a different way.

the inclusions $D \subset \bigcap_0^\infty \overline{O}_{n+1} \subset \bigcap_0^\infty O_n = D$, $\overline{O}_1 \in Q$ imply that D is an element of P with $A \subset D \subset B$. Since every open neighborhood B of an arbitrary point x contains a neighborhood $A \in Q$ of x, this proves the lemma.

THEOREM 10. If P is the family of all closed compact G_δ-sets of a locally compact general topological group, then the class K of all σ-finite left or right 0 invariant measures defined on P is harmonic as well as all of its perfections.

PROOF: Let $S = \overline{\overline{P}}$. The left regularity condition (i) for P being obviously satisfied, it is enough to apply lemma 1 with $G^* = G$ and lemma 2 in order to see that P is left regular relative to S. Then theorem 7 and symmetry reasons complete the proof.

REMARK. Theorem 10 combined with the property of Kakutani-Kodaira of locally compact topological groups yields another proof of theorem 9.

Finally an application of theorem 7 can be based on the following *covering lemma* for abstract groups which will be also used in another part of this paper.

LEMMA 3. Let X and Y be subsets of an abstract group G and assume that X has the following *Souslin property relative to* Y: *Every disjoint family of sets of the form yX with y in Y is at most denumerable.* Then Y can be covered by an at most denumerable family of sets of the form yXX^{-1} with y in Y.

PROOF: Assume the set Y to be well ordered in a possibly transfinite sequence of distinct elements y_α. We define A to be the subset of Y which contains the first element y_1 of Y and which contains an arbitrary element y_β of Y if and only if y_β is not an element of any set of the form $y_\alpha XX^{-1}$ with y_α in A and with $\alpha < \beta$.

Then, whenever y_α and y_β are elements of A with $\alpha < \beta$, the sets $y_\alpha X$, $y_\beta X$ are disjoint. Hence, by the Souslin property of X relative to Y, the set A is at most denumerable.

Moreover, every element y of A being contained in yXX^{-1} and every element y_β of $Y - A$ being contained in some set of the form $y_\alpha XX^{-1}$ with y_α in A, $\alpha < \beta$, the set Y is contained in AXX^{-1}, which completes the proof.

LEMMA 4. Let G be a general topological group satisfying the first axiom of countability. If a σ-finite left 0-invariant measure m is defined for all Borel sets of G and is positive on all open nonvoid sets, then every subset of G has the Lindelöf property.

PROOF: For an arbitrary representation of G as the union of a disjoint sequence $\{E_r\}$ of sets of finite m-measure and for any x in E_r let $f(x)=2^{-r}m(E_r)^{-1}$ if $m(E_r)$ is positive and let $f(x)=0$ otherwise. Then the indefinite integral of $f(x)$ with respect to m is a finite measure n equivalent with m. Since n is positive on every open set, any family of disjoint left translates of such a set is at most denumerable, i. e., such a set has the Souslin property relative to every subset of G.

Let $\{V_n\}$ be a countable base of the group unity such that the sets V_n are open and such that we have

$$V_{n+1}\cdot V_{n+1}^{-1}\subset V_n\,. \tag{6}$$

If F is an arbitrary subset of G and if Γ is a family of open sets whose union contains F, then let F_n be the set of all points x in F such that there is a set $0_n(x)$ in Γ which contains xV_n. Clearly F is contained in the union of the sets F_n. On the other hand, since the set V_{n+1} has the Souslin property relative to F_n, there is, by lemma 3, a sequence of elements $x_{in}\,(i=1,2,\cdots)$ of F_n such that F_n is contained in the union of the sets $x_{in}V_{n+1}V_{n+1}^{-1}$. This implies, in view of (6), that F_n is contained in $\bigcup_i 0_n(x_{in})$, hence F is a subset of $\bigcup_n\bigcup_i 0_n(x_{in})$, which completes the proof.

THEOREM 11. Let G be a general topological group which satisfies the first axiom of countability. If the class of all σ-finite either left or right 0-invariant measures defined on the σ-field S of all Borel sets of G contains a measure which is positive on all sets with nonvoid interior, then this class is harmonic.

PROOF: The σ-field S being generated by the family P of all closed subsets of G, it is sufficient to show that the family P is both left and right regular relative to S. Now it is well known that P contains a complete system of neighborhoods of every point in G and that it is left invariant, consists of G_δ-sets and satisfies the left regularity condition (i). By lemma 4 the elements of $P\times P$ have the Lindelöf property in $G\times G$ (since this topological product satisfies the first axiom of contability), hence, by lemma 1 with $G^*=G$, the family P is left regular relative to S. Theorem 7 and symmetry reasons complete the proof.

COROLLARY. The class of all σ-finite either left or right 0-invariant measures defined on the σ-field of all Borel sets of a general topological group satisfying the second axiom of countability is harmonic.

5. Determination of atomless measures. To conclude this chapter with an easy application of theorem 3, we return to the assumptions made in section 2 and we call the family of all sets on which a measure m assumes a fixed finite value c, a *level family* or, more specifically, the *c-family* of m. The problem arises under what conditions a measure is uniquely determined by a class L of its level families. To formulate such a condition, we define: A class L of families of subsets of the set G is *bounded* if it contains a family of sets the elements of which contain every element of any other family in L.

THEOREM 12. An atomless measure is uniquely determined by every infinite and bounded class L of its level familes.

PROOF: Let ARB mean that A and B are sets which belong to the same element of L. Let K be the class of all atomless measures which admit all elements of L as level families. First we prove theorem 12 for the special case that L contains an infinite sequence $\{F_k\}$ of elements on which every measure in K assumes positive values converging to 0. In this case every measure m in K has the Intersection Property modulo R. For it is well known that a finite atomless measure assumes all intermediate values; therefore, to any two sets A, B in S_m there is a level family in $\{F_k\}$ which contains sets $A' \subset A$, $B' \subset B$. The class K being closed under addition, and all of its elements having the variation 1 under R, theorem 3 yields the assertion.

Next we reduce the general case to this special one as follows. We choose both an arbitrary measure m in K and an arbitrary strictly monotone sequence $\{c_k\}$ of values of m on the elements of L. The class L being bounded, the sequence $\{c_k\}$ converges. Let c_0 be its limit and let $\tilde{c}$ denote the c-family of m for every non-negative number c. If the sequence $\{c_k\}$ is decreasing, then it is easily verified that: (i) $\tilde{c}_0$ is the family of all intersections of descending sequences $\{A_k\}$ of sets $A_k \in \tilde{c}_k$; (ii) if $d_k = c_k - c_0$, then $\tilde{d}_k$ is the family of all sets of the form $A_k - A_0$, where A_0 is an arbitrary element of $\tilde{c}_0$ and A_k is an arbitrary element of $\tilde{c}_k$ which contains A_0 $(k = 1, 2, \cdots)$.

Since (ii) implies that the families $\tilde{d}_k$ are level families for every measure in K, they can be included in L and chosen to be the sequence $\{F_k\}$ of the special case. If $\{c_k\}$ is increasing, then the families $\tilde{d}_k$ with $d_k = c_0 - c_k$ $(k = 1, 2, \cdots)$ can be verified to have the same property. In either case, L can be extended so as to satisfy the conditions of the special case.

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