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NOTE ON A THEOREM ON ABSTRACT DIFFERENTIAL EQUATIONS*

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In a recent paper [1], A. S. Amitsur proved the following theorem: If F is a field, not necessarily commutative, S is an automorphism of F, D is an additive function on F to F such that

$$\mathrm{D}(x \cdot y) = \mathrm{S}(x) \cdot \mathrm{D}(y) + \mathrm{D}(x) \cdot y \tag{1}$$

and C is the subfield of elements x such that $\mathrm{D}(x)=0$; then the elements of F which are solutions of the equation

$$a_0 z^{(n)} + \cdots + a_n z = 0, \tag{2}$$

where $z^{(k)} = \mathrm{D}^{(k)}(z)$ form a right C-module of dimension at most n. The proof of this theorem was a generalization of a familiar argument involving the Wronskian.

The purpose of this note is to point out that this theorem can be proved by the method suggested in another note [2] for the corresponding theorem in undergraduate differential equations.

To this end we define a linear differential operator inductively as an operator $\mathrm{L}(x)$ over F satisfying one of the following four conditions (*a*) $\mathrm{L}(x)=ax$ where a is any element of F; (*b*) $\mathrm{L}(x)=\mathrm{D}(x)$; (*c*) $\mathrm{L}(x)=\mathrm{L}_1(x)+\mathrm{L}_2(x)$, and (*d*) $\mathrm{L}(x)=\mathrm{L}_1(\mathrm{L}_2(x))$. In the cases (*c*) and (*d*) $\mathrm{L}_1(x)$ and $\mathrm{L}_2(x)$ are operators already known to be linear differential operators. Then it is easily shown that every linear differential operator is equivalent (by (1)) to an L such that $\mathrm{L}(z)$ is of the form shown on the left in (2). In that case L will be said to be of order n.

* Received January, 1951.

(1) A generalization of a theorem on linear differential equations, *Bull. Amer. Math. Soc.*, **54**, pp. 937-941 (1948).

(2) Certain basic theorems in linear differential equations. *Amer. Math. Monthly*, **56**, pp. 398-402 (1949).

With this definition it can be shown that for any such L

$$\mathrm{L}(x \cdot y) = \mathrm{L}(x) \cdot y + \mathrm{M}(x\,;\,y'),$$

where for any given x, $\mathrm{M}(x\,;u)$ is a linear differential operator on u which vanishes identically when L is of order zero and is of order $n-1$ when L is of order $n>0$. This is accomplished by induction using the above properties (a) to (d). The verification of the statement about the order is not difficult provided we use the form (2), since (c) is used only when L_1 and L_2 are of different orders, and (d) only when L_1 is D or $\mathrm{L}_1(x)=ax$.

This established, the proof of Amitsur's theorem follows by the same method as in my paper above cited.

We can show further that the dimension is exactly n provided (2) always has a non trivial solution x_1, and the equation $\mathrm{D}(x)=y$ has at least one solution x for any y. For then $x=x_1 y$ is a solution of (2) if and only if $\mathrm{M}(x_1, y')=0$. By the hypothesis of the induction, this equation has a basis $v_2, v_3, \cdots, v_n$; let y_i be a solution of $y'_i = v_i$. Then $x_1, x_2=x_1 y_2, \cdots, x_n=x_1 y_n$ form a basis for (2). (For $n=1$, $\mathrm{M}=0$ has no non trivial solutions).