PORTUGALIAE MATHEMATICA

ISSN 0032-5155

VOLUME 43

1985-1986

Edição da

SOCIEDADE PORTUGUESA DE MATEMÁTICA

PORTUGALIAE MATHEMATICA

Av. da República, 37-4.º

1000 LISBOA—PORTUGAL

PORTUGALIAE MATHEMATICA
Vol. 43 — Fasc. 1 — 1985-1986

ON POLYNOMIAL INVARIANT CHAINS OF MATRICES AND PRINCIPAL SUBMATRICES

BY

JOÃO CARLOS DAVID VIEIRA
Departamento de Matemática, Universidade de Aveiro
3800 Aveiro — PORTUGAL

ABSTRACT. Let $\chi_{s_1}, \ldots, \chi_{s_q}$ *and* $\xi_{r_1}, \ldots, \xi_{r_t}$ *be two sequences of nonzero monic polynomials with* s_j *and* r_i *integer numbers satisfying the inequalities* $1 \leqslant s_1 < \ldots < s_q < n$ *and* $1 \leqslant r_1 < \ldots < r_t < n + p$. *Necessary and sufficient conditions are given for matrices* A *and* B *to exist such that:* (i) B *is an* $n + p$ *square matrix having* A *as principal* n*-square block;* (ii) *the characteristic invariant chains of* A *and* B *(namely* $\alpha_1, \ldots, \alpha_n$ *and* $\beta_1, \ldots, \beta_{n+p}$*) satisfy* $\alpha_{s_j} = \chi_{s_j}$ *and* $\beta_{r_i} = \xi_{r_i}$, *for* $j = 1, \ldots, q$ *and* $i = 1, \ldots, t$.

Note that no minimal polynomial is prescribed.

1. Introduction

In this paper **K** denotes an arbitrary field and greek letters are used to represent either monic polynomials in the variable λ or polynomial chains. A and B represent square matrices over **K** of orders respectively n and $n + p$ ($n \geqslant 1$ and $p \geqslant 0$). Given two polynomials μ and ν we write $\mu <: \nu$ to mean that μ *divides* ν. Sometimes we will use the following notation

$$\inf(\mu, \nu) := \gcd(\mu, \nu) \quad \text{and} \quad \sup(\mu, \nu) := \text{lcm}(\mu, \nu),$$

where gcd and lcm stand for *greatest common divisor* and *least common multiple*.

Received December 5, 1984. Revised April, 1 1986.

Recall that the *characteristic invariant factors* of A are the invariant factors of $\lambda I - A$ which we represent by $\alpha_1, \ldots, \alpha_n$ and are defined by

$$\alpha_1 \alpha_2 \ldots \alpha_k = \gcd \{\text{minors of order } k \text{ of } \lambda I - A\},$$

for $k = 1, \ldots, n$. Obviously $\alpha_1 <: \alpha_2 <: \ldots <: \alpha_n$. Similarly are defined the characteristic invariants of B, denoted by $\beta_1, \ldots, \beta_{n+p}$.

The sequence $\alpha := (\alpha_1, \ldots, \alpha_n)$ is called the *characteristic invariant chain of* A, or simply the *invariant chain of* A.

Following [1] we also denote by α the infinite sequence of invariant factors $(\ldots, 1, 1, \alpha_1, \alpha_2, \ldots, \alpha_n, 0, 0, \ldots)$ obtained from $(\alpha_1, \ldots, \alpha_n)$ by adding to it an initial infinite string of 1's and a final infinite string of 0's. Therefore we have $\alpha = (\alpha_i : i \in \mathbf{Z})$, where

$$\alpha_i = 1, \text{ if } i < 1 \; ; \; \alpha_i = 0, \text{ if } i > n \; ; \; \alpha_i <: \alpha_{i+1}, \text{ for } i \in \mathbf{Z}.$$

Similarly, we name by *invariant chain of* B either the sequence $(\beta_1, \ldots, \beta_{n+p})$ or the infinite sequence $\beta = (\beta_i : i \in \mathbf{Z})$, where $\beta_i = 1$ if $i < 1$, and $\beta_i = 0$ if $i > n + p$.

Generally speaking, a *chain* is a sequence of polynomials $\gamma = (\gamma_i : i \in \mathbf{Z})$, such that $\gamma_i <: \gamma_{i+1}$. The *rank* of a chain γ is defined by

$$\text{rank}(\gamma) := \inf \{i : \gamma_{i+1} = 0\}.$$

For a given integer k and a given chain γ with rank $(\gamma) \geq k$, we define the k – *degree* of γ by

$$\deg_k(\gamma) := \sum_{i \leq k} \deg(\gamma_i)$$

A matrix A is said to be c – *imbeddable in* B if A is a principal submatrix of a matrix $\overline{B}$ similar to B.

Let us fix once for all the following two sequences of nonzero monic polynomials to be used in what follows:

$$(1.1) \qquad \chi_{s_1}, \ldots, \chi_{s_q} \text{ and } \xi_{r_1}, \ldots, \xi_{r_t},$$

where the s_j and r_i are fixed integers such that $1 \leq s_1 < \ldots < s_q < n$, $1 \leq r_1 < \ldots < r_t < n + p$. We always assume that

$$\chi_{s_1} <: \ldots <: \chi_{s_q} \text{ and } \xi_{r_1} <: \ldots <: \xi_{r_t}.$$

2. Some sets of chains

Let us denote by C the set of all chains

$$C := \{\gamma = (\gamma_i) : \gamma_i <: \gamma_{i+1} \text{ for all } i \in \mathbf{Z}\},$$

and define on C, in a natural way, the following partial ordering:

$$\gamma <: \delta \quad \text{iff} \quad \gamma_i <: \delta_i \quad \text{for all } i \in \mathbf{Z},$$

and the two operations

$$\inf(\gamma, \delta) := (\gcd(\gamma_i, \delta_i) : i \in \mathbf{Z}), \quad \sup(\gamma, \delta) := (\mathrm{lcm}(\gamma_i, \delta_i) : i \in \mathbf{Z}).$$

With these two operations, C is obviously a complete lattice.
In C define the *shifting operator of order* s, for any integer s, by the following convention:

$E^s \gamma$ *is the chain* γ' *with coordinates* $\gamma_i' := \gamma_{i+s}$.

Consider the following subset of C

$$C_\chi := \{\gamma \in C : \gamma_{s_j} = \chi_{s_j} \text{ for } j = 1, \ldots, q\}$$

where χ_{s_j} are the polynomials referred in (1.1). It is easily seen that, given an arbitrary set of chains of C_χ, the infimum and the supremum of this set also belong to C_χ. Thus C_χ is a complete sublattice of C.
Let χ^m and χ^M be the chains whose coordinates are given by

$$\chi_i^m := \begin{cases} 1 & \text{if } \quad i < s_1 \\ \chi_{s_j} & \text{if } s_j \leqslant i < s_{j+1} \\ \chi_{s_q} & \text{if } s_q \leqslant i \end{cases} \qquad \chi_i^M := \begin{cases} \chi_{s_1} & \text{if } \quad i \leqslant s_1 \\ \chi_{s_k} & \text{if } s_{k-1} < i \leqslant s_k \\ 0 & \text{if } \quad s_q < i \end{cases}$$

It is easy to prove the following:

PROPOSITION 2.1. *The chain* χ^m *is the infimum of* C_χ *and* χ^M *is the supremum of* C_χ.

A similar result can be proved for the set

$$C_\xi := \{\delta \in C : \delta_{r_k} = \xi_{r_k}, \text{ for } k = 1, \dots, t\}.$$

As a matter of fact, if we define ξ^m and ξ^M by

$$\xi_i^m := \begin{cases} 1 & \text{if } i < r_1 \\ \xi_{r_k} & \text{if } r_k \leqslant i < r_{k+1} \\ \xi_{r_t} & \text{if } r_t \leqslant i \end{cases} \qquad \xi_i^M := \begin{cases} \xi_{r_1} & \text{if } i \leqslant r_1 \\ \xi_{r_k} & \text{if } r_{k-1} < i \leqslant r_k \\ 0 & \text{if } r_t < i \end{cases}$$

we have the following

PROPOSITION 2.2. *The chain ξ^m is the infimum of C_ξ and ξ^M is the supremum of C_ξ.*

The following set $P(\chi, \xi)$ will play an important role in the sequel (recall that p is a fixed nonnegative integer):

$$P(\chi, \xi) := \{(\gamma, \delta) : \delta <: \gamma <: E^{2p}\,\delta, \gamma \in C_\chi, \delta \in C_\xi\}.$$

THEOREM 2.3. *The set $P(\chi, \xi)$ is nonempty if and only if the following relations hold:*

$$\xi^m <: \chi^M \text{ and } \chi^m <: E^{2p}\,\xi^M. \tag{2.1}$$

Proof. Assume that there exists a pair (γ, δ) in the set $P(\chi, \xi)$. Then it follows that

$$\chi^m <: \gamma <: \chi^M, \xi^m <: \delta <: \xi^M \text{ and } \delta <: \gamma <: E^{2p}\,\delta.$$

From these relations we obtain (2.1) by transitivity.

Conversely, assume (2.1) holds. Consider the chains γ and δ given by $\gamma := \sup(\chi^m, \xi^m)$ and $\delta := \sup(\xi^m, E^{-2p}\,\chi^m)$. Using (2.1) it can be proved that (γ, δ) belongs to $P(\chi, \xi)$. We omit the details. ∎

In the sequel we shall use the following additional notations:

$$\begin{aligned} \chi^\sigma &:= \inf\ (\chi^M, E^{2p}\,\xi^M) & \xi^\sigma &:= \inf\ (\xi^M, \chi^M) \\ \chi^\iota &:= \sup\ (\chi^m, \xi^m) & \xi^\iota &:= \sup\ (\xi^m, E^{-2p}\,\chi^m). \end{aligned} \tag{2.2}$$

In a natural way we define in $P(\chi, \xi)$ a partial order relation by

$$(\alpha, \beta) <: (\gamma, \delta) \text{ iff } \alpha <: \gamma \text{ and } \beta <: \delta.$$

It can be proved that $P(\chi, \xi)$ is a complete sub-lattice of $C_\chi \times C_\xi$ with respect to this partial ordering. In particular the supremum and the infimum of $P(\chi, \xi)$ may be described in terms of the chains defined in (2.2). This is the purpose of the following theorem whose proof we omit.

THEOREM 2.4. *Assume $P(\chi, \xi)$ is nonempty. Then $(\chi^\sigma, \xi^\sigma)$ and (χ^ι, ξ^ι) are respectively the supremum and the infimum of $P(\chi, \xi)$.*

3. Main results

THEOREM 3.1. *Let* A *and* B *be two square matrices of orders* n *and* n+p *with characteristic invariant chains* α *and* β, *respectively. If* $\alpha \in C_\chi$, $\beta \in C_\xi$ *and if* A *is* c-*imbeddable in* B, *then we have*

$$\chi^m <: E^{2p}\, \xi^M \text{ and } \xi^M <: \chi^m. \tag{3.1}$$

Proof. From the hypotheses and from the results in [1, 2] it follows that the pair (α, β) belongs to $P(\chi, \xi)$. Therefore, (3.1) follows from Theorem 2.3. ■

THEOREM 3.2. *Under the conditions of Theorem 3.1 we have the following degree inequalities:*

$$\deg_n(\chi^\iota) \leqslant n \text{ and } \deg_{n+p}(\xi^\iota) \leqslant n + p. \tag{3.2}$$

Proof. From the hypotheses it follows that rank $(\alpha) = \deg_n(\alpha) = n$ and rank $(\beta) = \deg_{n+p}(\beta) = n + p$. Since (α, β) belongs to $P(\chi, \xi)$ it follows from Theorem 2.4 that $\chi^\iota <: \alpha$ and $\xi^\iota <: \beta$. This yields (3.2). ■

THEOREM 3.3. *Assume that the sequences* χ *and* ξ *satisfy the conditions* (3.1) *and* (3.2). *Then two matrices* A *and* B *exist such that* A *is* c-*imbeddable in* B, *the characteristic invariant chain of* A *belongs to* C_χ *and the characteristic invariant chain of* B *belongs to* C_ξ.

Proof. Conditions (3.1) mean that $P(\chi, \xi)$ is nonempty. We use the infimum of $P(\chi, \xi)$ given by (2.2) and Theorem 2.4 in the following construction of two chains α and β:

$$\alpha_i = \chi_i^{\iota} \text{ if } i < n; \qquad \alpha_n = \upsilon \chi_n^{\iota} \quad \text{and } \alpha_i = 0 \text{ if } i > n$$

$$\beta_j = \xi_j^{\iota} \text{ if } j < n + p; \quad \beta_{n+p} = \omega \xi_{n+p}^{\iota} \text{ and } \beta_j = 0 \text{ if } j > n + p.$$

In these definitions of α and β, υ and ω are any polynomials such that

$$\deg(\upsilon) = n - \deg_n(\chi^{\iota}) \quad \text{and} \quad \deg(\omega) = n + p - \deg_{n+p}(\xi^{\iota}).$$

Such polynomials exist because of (3.2). It is a simple matter to prove that the pair (α, β) belongs to $P(\chi, \xi)$. The results in [1, 2] may again be applied to conclude the proof. ∎

REFERENCES

1. E. Marques de Sá — *Imersão de Matrizes e Entrelaçamento de Factores Invariantes,* Tese de Doutoramento, Universidade de Coimbra, 1979.
2. R. C. Thompson — *Interlacing Inequalities for Invariant Factors,* Linear Algebra and its Applications, 24 (1979), 1-32.
3. J. David Vieira — *Partial Prescription of Matrices and Submatrices Invariant Polynomials,* Algebra Lineal y Aplicaciones, Actas del Encuentro Internacional de Algebra Lineal y Aplicaciones, Vitoria, 1983, pp 418-424