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RESTRICTED SEPARATION OF POLYHEDRA *

BY D. G. BOURGIN
University of Illinois, Urbana, Illinois, U. S. A.

This note presents some natural separation properties for special convex polyhedra. The results are quite elementary and would seem to intervene as basic lemmas in a number of problems.

Let $E = \{e_i \mid i = 1, \cdots, n\}$ be an $n-1$ dimensional open fixed reference simplex in the Euclidean space R^{n-1}. In the sequel E is always assumed replaced by a properly containing simplex, if necessary, compassing all bounded sets under consideration. Let t designate the point of R^{n-1} whose barycentric coordinates with respect to $\{e_i\}$ are $\{t_i\}$. Thus $\Sigma t_i = 1$ and if $t \in E$ then $0 < t_i < 1$. Let I_i designate the open unit interval with parameter t_i, let N_i be an open interval of I_i. Then $G = (\Pi N_i) \cap E$ is an open set of E. We say $\{N_i \mid i = 1, \cdots n\}$ is minimal for G if $G = \Pi N_i' \cap E$ implies $N_i \subset N_i'$. The designation *hyperplane*, $H(i)$, below invariably denotes a locus $t_i = c_i$, c_i a positive constant inferior to 1 so $H(i)$ cuts E. Let the hyperplanes $\{H(i) \mid i = 1, \cdots n\}$ determine an open simplex, σ. We denote the vertex of σ opposite $H(i)$ by $v(i)$ and the face opposite $v(i)$ by $F(i)$. Thus $F(i) \subset H(i)$. Since $H(i)$ cuts E, either e_i and $v(i)$ are separated by, or are on the same side of $H(i)$.

Consider the latter case. Bring $v(i)$ to e_i by a translation. It is then evident that the faces of σ containing $v(i)$ now coincide with corresponding faces of E containing e_i in a half space determined by $H(i)$. Accordingly $v(j)$ is on the same side of $H(j)$ as e_j for all j. If then $v(i)$ and e_i are on opposite sides of $H(i)$ for one value of i, this separation maintains for all i. In the first case we say σ is *inverse* and in the second that σ is *direct*. In the inverse case e_i may be a point of $H(i)$. If σ is inverse and σ' is direct we say σ and σ' are paired (with respect to E).

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THEOREM 1. *The disjunct paired simplexes σ and σ' can be separated by a hyperplane parallel to one of the faces.*

This suggests the general problem for arbitrary n and r where the two n dimensional polyhedra have their faces parallel to r fixed planes.

Recall our special usage of «hyperplane» demands *parallelism to a face.* The hyperplane $H(i)$ sections R^{n-1} into the open half spaces: $H(i)^+$ containing σ and $H(i)^-$. The condition that $H(i)$ be a separating plane is of course that $H(i)^- \supset \sigma'$. Suppose (I) $H(i) \cap \sigma' = \phi$ independently of i. It is impossible that $H(i)^+ \supset \sigma'$ for all i for then $\bigcap H(i)^+ \supset \sigma'$. The left hand side is evidently σ so that we have reached a contradiction. Thus the theorem's assertion is valid in case (1).

Suppose now (II): $H(i) \cap \sigma' = \phi$, $i \in M_1$; $H(i) \cap \sigma' \neq \phi$, $i \in M_2$ where M_2 is a non empty set and $M_1 \cup M_2 = \{i \mid i = 1, \cdots n\} = M$. The non trivial case is that for which $H(i)^+ \supset \sigma'$, $i \in M_1$. In this case we consider the section by $H(i)$ for some fixed i in M_2. By reordering we may take this i as 1 and write $\sigma(1)$, $\sigma'(1)$ for $F(1)$ and $H(1) \cap \sigma'$ respectively. Similarly $E \cap H(1)$ is $E(1)$. It is easy to see that $\sigma(1)$ and $\sigma'(1)$ are paired simplexes with respect to $E(1)$. Write $H(1, j)$ for $H(1) \cap H(j)$. Then $H(1, j) \cap \sigma'(1) = H(1) \cap H(j) \cap \sigma' = \phi$ for $j \in M_1$. Moreover

$$H(1, j)^+ = H(j)^+ \cap H(1) \supset \sigma' \cap H(1) = \sigma'(1), j \in M_1 .$$

We remark that if $j \in M_2 - (1)$ then there are two cases, namely

(a) $$H(1, j) \cap \sigma'(1) = \phi, j \in L_1,$$

(b) $$H(1, j) \cap \sigma'(1) \neq \phi, j \in L_2,$$

where $L_1 \cup L_2 = M_2 - (1)$.

Note $H(1) \cap \sigma' \neq \phi$ implies $H(1) \cap F'(j) \neq \phi$ for $j \neq 1$. Observe that if $H(j) \cap \sigma' \neq \phi$ then $v'(j)$ and $v(j)$ are separated by $H(j)$ since σ and σ' are paired. Hence $H(j)^+ \supset F'(j)$ for $j \in M_2 - (1)$. Thus $H(j)^+ \cap H(1) \supset F'(j) \cap H(1) \neq \phi$. Accordingly if $j \in L_1$, we have $H(1, j)^+ \supset \sigma'(1)$.

We have shown therefore that if there is no separating hyperplane originally there can be none in the section gotten by intersecting with $H(i)$, which reduces the situation to one of dimension 1 less. This reduction replaces the finite set M_2 by a proper subset L_2. If $L_2 \neq \phi$, consider the sections of $E(1)$, $\sigma(1)$, $\sigma'(1)$ by $H(j)$ for some $j \in L_2$. Proceed till we arrive at the situation for case (I) for a subspace of dimension at least 1. If $H(i)^+ \supset \sigma'$ for all $i \in M_1$ our analysis has shown that there can be no separating hyperplane in any of the sections. In view of the situation in case 1 this can be the case only if ultimately there is an inclusion relation between the sections of σ and σ'. Thus in case II, $H(i)^{-}\supset\sigma'$ for some $i\in M_1$.

For proper perspective we remark that if σ and σ' are two direct or two inverse disjunct simplexes Theorem 1 is no longer necessarily valid and there need be no separating plane parallel to a face of E if $n\geqq 4$. For an example in R^3 let σ be determined by the vertex system $1,0,0;-1,0,0;0,1,1$ and $0,-1,1$. Let σ' be obtained by translating σ in the z direction by an amount $1+\varepsilon$, $\varepsilon\geqq 0$. If ε is sufficiently small it is easy to verify σ,σ' furnish the example sought. On taking their join with a point not in this R^3 it may be seen an example in R^4 is afforded, etc.

THEOREM 2. *If* G *and* G′ *are non trivial open sets in* E *of the form* $\Pi N_i\cap E$, $\Pi N_i'\cap E$ *then, if the neighborhoods for* G *and for* G′ *are minimal,* $G\cap G'=\emptyset$ *implies and is implied by* $N_i\cap N_i'=\emptyset$ *for at least one* i.

The assertion of the theorem may be invalid if the neighborhoods for one of G or G′ are not minimal. Let $\cdot N_l=N_l\times\Pi_{i\neq l}I_i$.

Since sufficiency is obvious we need only prove necessity. Let $N_i=\{t_i\mid a_i<t_i<b_i\}$, $N_i'=\{t_i\mid A_i<t_i<B_i\}$ where $A_i<b_i$, $a_i<B_i$. Suppose $H(l)\subset\cdot N_l\cap\cdot N'_l$. Let S_1 be the open simplex in R^{n-1} determined by the hyperplanes $t_i=a_i$, with vertices $\alpha(i)$ and faces $g(i)$ opposite these vertices. Similarly let S_2 be the open simplex defined by the choices $t_i=b_i$, with vertices and opposing faces denoted by $\beta(i)$ and $f(i)$ respectively. Then S_1 and S_2 are paired since $G(=E\cap S_1\cap S_2)\neq\emptyset$. By slight change in $\{c_i\}$, if necessary, we ensure there is an open simplex σ determined by $\{H(i)\}$ with the associated nomenclature used earlier.

The minimality hypothesis implies

$$\text{(a)}\qquad H(i)\cap G\neq\emptyset,\quad H(i)\cap G'\neq\emptyset$$

Indeed if $H(l)\cap G=\emptyset$, then $H(l)$ sections N_l into two disjunct sub neihborhoods N_l^1 and N_l^2 with $\cdot N_l^2\cap G=\emptyset$. Accordingly N_l^1 could replace N_l^2 in defining G. Similarly $H(l)$ cuts G′.

Clearly σ is interior either to S_1 or to S_2. Suppose therefore S_1 contains σ. We assert now that (1) G *intersects* each face of σ. If $\sigma\subset S_1\cap S_2$ this assertion is trivially satisfied. Observe that if σ is paired with S_1 it must be contained in S_2. Hence the non trivial case for the assertion I is that in which S_1 contains σ and S_2 does not, and σ is paired with S_2. Then $\alpha(l)$ and $v(l)$ are on the same side of $H(l)$. (The ensuing arguments are similar for S_1 and S_2 interchanged). $H(l)$ divides N_l into N_l^1 and N_l^2 where $N_l^1 = \{t_l \mid t_l > c_l\} \cap N_l$. Evidently $E \cap \Pi N_l^2 = \emptyset$. Suppose $E \cap \Pi N_l^1 = \emptyset$ also. Then with $n_i(j) \in N_i^j$, $j = 1, 2$, it is easy to see $\Sigma n_i(1) > 1 > \Sigma n_i(2)$ whence it follows that $\Sigma c_i = 1$, $i \cdot e \cdot \sigma$ is empty. This contradiction establishes $E \cap \Pi N_l^1 \neq \emptyset$. Actually since any point outside σ is separated from some N_l^1 by an $H(l)$ we have $\sigma \supset E \cap \Pi N_l^1$, and so

(b) $$G \cap \sigma \neq \emptyset.$$

Suppose $F(i)$ does not intersect G. Since S_1 contains σ it follows that $F(i) \cap S_2 = \emptyset$. On the other hand (a) implies $H(i) \cap S_2 \neq \emptyset$. Theorem 1 implies $F(i) \cap F(k)$ separates $F(i)$ from $H(i) \cap S_2$ so $H(k)$ separates σ from S_2 in violation of (b). Thus assertion I is substantiated.

We establish II: *There is a hyperplane* $h(l)$ *parallel to* $H(l)$ *such that* $G \supset h(l) \cap \sigma \neq \emptyset$. If $v(l) \in S_2$ some neighborhood of $v(l)$ must be in S_2 and hence an $h(l)$ satisfying (II) exists. If $v(l) \bar{\in} S_2$, $v(l)$ is separated from $H(l)$ by $t_l = b_l$, which by the minimality hypothesis cuts S_1 in a set contained in $f(l)$. Then, for ε sufficiently small, $h(l)$ given by $t_l = b_l - \varepsilon$ satisfies (II). Similar assertions can be made for G' and σ. Actually one l value only is needed below.

Let $K(l)$ and $K'(l)$ be the sections of σ by hyperplanes $h(l)$ and $h'(l)$ which satisfy (II) for G and G' respectively. If $K(l) = K'(l)$ the common points are in $G \cap G'$. If $K(l) \cap K'(l) = \emptyset$ then one of these sections separates the other from the face $F(l)$. Suppose then $K'(l)$ is between $K(l)$ and $F(l)$. Since (I) guarantees G contains a point x on $F(l)$, the convex hull of $K(l)$ and x, which is in G by the convexity of G, must intersect $K'(l)$. The intersection is in $G \cap G'$. The theorem is therefore established.