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REMARK ON HOMOTOPY INVERSES *

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1. Assume that for each space X of a given collection $\mathfrak{A}$ there is given a group $G(X)$, and that for each (continuous) map $f: X \to Y$ there is given a homomorphism $h(f, G): G(X) \to G(Y)$ depending only on the homotopy class of f, such that (1): if $f: X \to Y$ and $g: Y \to Z$ then $h(gf, G) = h(g, G)h(f, G)$ and (2): the identity map of each X is associated with the identity automorphism of $G(X)$(1). The collection $\{G(X), h(f, g)\}$ is called an h-system on $\mathfrak{A}$(2). Given two h-systems $\{G_1(X), h(f, G_1)\}$, $\{G_2(X), h(f, G_2)\}$ on $\mathfrak{A}$ and a definite $Y \in \mathfrak{A}$, a homomorphism $\eta: G_1(Y) \to G_2(Y)$ is natural if for each $f: Y \to Y$ the diagram

$$\begin{array}{ccc} G_1(Y) & \xrightarrow{h(f, G_1)} & G_1(Y) \\ \eta \downarrow & & \downarrow \eta \\ G_2(Y) & \xrightarrow[h(f, G_2)]{} & G_2(Y) \end{array}$$

is commutative.

Restrict attention to one h-system on $\mathfrak{A}$ and a definite $f: X \to Y$. If f is a homotopy equivalence [4; 1133], then $h(f, G): G(X) \approx G(Y)$. If f has only a left homotopy inverse(3), then $h(f, G)$ is an isomorphism of $G(X)$ into $G(Y)$ (and onto a direct summand if $G(Y)$ is abelian). The object of this note is to remark that if one considers two h-systems on $\mathfrak{A}$ something more may be said in this latter case, and to give a simple application.

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(1) In other terms, G is a functor defined on the category (X, α) where $X \in \mathfrak{A}$ and α is a homotopy class of maps, with values in the category of groups.

(2) The concept simply lists the properties in the category $\mathfrak{A}$ of say, the homotopy groups, or the singular homology groups, or the Čech groups based on all coverings, that are required in the simple proof of **1. 1**. Note that Čech groups based on finite coverings in non-compact spaces do not form an h-system.

(3) That is, there is a $g: Y \to X$ with $gf \simeq$ identity.

In fact,

1.1 Theorem. *Let* $\{G_1(X), h(f, G_1)\}$, $\{G_2(X), h(f, G_2)\}$ *be two* h*-systems on* $\mathfrak{A}$. *Assume that at* $Y \in \mathfrak{A}$ *there is a natural isomorphism* $\eta : G_1(Y) \approx G_2(Y)$. *If for any* $X \in \mathfrak{A}$ *there are maps*

$$X \xrightarrow{f} Y \xrightarrow{g} X$$

with $gf \simeq$ *identity, then* $G_1(X)$ *is isomorphic to* $G_2(X)$ [4].

Proof. Setting $X_i = G_i(X), Y_i = G_i(Y), f_i = h(f, G_i), g_i = h(g, G_i)$, $i = 1, 2$, one has the diagram

$$\begin{array}{ccccccc} X_1 & \xrightarrow{f_1} & Y_1 & \xrightarrow{g_1} & X_1 & \xrightarrow{f_1} & Y_1 \\ & & \downarrow \eta & & & & \downarrow \eta \\ X_2 & \xrightarrow[f_2]{} & Y_2 & \xrightarrow[g_2]{} & X_2 & \xrightarrow[f_2]{} & Y_2 \end{array} \tag{1}$$

Since the f_i are isomorphisms into, and the η is an isomorphism onto, it suffices to show $\eta[\text{Image } f_1] = \text{Image } f_2$. If $y_1 \in \text{Image } f_1$ then $y_1 = f_1(x_1)$ for some $x_1 \in X_1$ so that $\eta(y_1) = \eta f_1 g_1 f_1(x_1) = f_2 g_2 \eta f_1(x_1)$ showing $\eta(y_1) \in \text{Image } f_2$. Conversely, if $\eta(y_1) \in \text{Image } f_2$, then $\eta(y_1) = f_2(x_2)$ for some $x_2 \in X_2$ and $\eta[f_1 g_1(y_1)] = f_2 g_2 \eta(y_1)$ $= f_2 g_2 f_2(x_2) = \eta(y_1)$ so $y_1 = f_1 g_1(y_1)$ and $y_1 \in \text{Image } f_1$.

1.2 Remark. The isomorphism $\bar{\eta} = f_2^{-1} \eta f_1 : X_1 \approx X_2$ in **1.1.** makes the diagram (1) commutative. From this follows: if there is an $\tilde{\eta} : G_1(X) \to G_2(X)$ which, for all $\varphi : X \to Y$ makes

$$\begin{array}{ccc} X_1 & \xrightarrow{h(\varphi, G_1)} & Y_1 \\ \tilde{\eta} \downarrow & & \downarrow \eta \\ X_2 & \xrightarrow[h(\varphi, G_2)]{} & Y_2 \end{array}$$

commutative, then under the hypotheses of **1.1**, $\tilde{\eta} : X_1 \approx X_2$.

2. Application. Recall [4; 1133] that a space P dominates a space X if and only if there are maps $\varkappa : X \to P$, $g : P \to X$ with $g\varkappa \simeq$ identity.

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(4) With no naturality condition on $\varkappa : G_1(Y) \approx G_2(Y)$, the case that $f : X \to Y$ has a right homotopy inverse leads trivially to the result: $G_1(X)/$ kernel $h(f, G_1)$ is isomorphic to $G_2(X)/$ kernel $h(f, G_2)$.

2.1 THEOREM. *In any space dominated by a* C W *polytope* (5), *the* CECH *homology groups based on all coverings* [1 ; 278-292] *and the singular homology groups* [3 ; 407-447] *are isomorphic in each dimension.*

PROOF. Since both these theories, with the usual induced homomorphisms form an h-system on, say, $\mathfrak{A} = \{X, P\}$ and since the two groups are isomorphic naturally in each dimension in polytopes, the result follows from **1.**1.

REMARK. Any metric space that is an absolute neighborhood retract for metric spaces is dominated by a C W polytope. [2 ; 365].

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(5) The C W topology on the polytope [5 ; 315-321] is taken for definiteness