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## REMARK ON HOMOTOPY INVERSES \*

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1. Assume that for each space  $X$  of a given collection  $\mathfrak{A}$  there is given a group  $G(X)$ , and that for each (continuous) map  $f: X \rightarrow Y$  there is given a homomorphism  $h(f, G): G(X) \rightarrow G(Y)$  depending only on the homotopy class of  $f$ , such that (1): if  $f: X \rightarrow Y$  and  $g: Y \rightarrow Z$  then  $h(gf, G) = h(g, G)h(f, G)$  and (2): the identity map of each  $X$  is associated with the identity automorphism of  $G(X)$ <sup>(1)</sup>. The collection  $\{G(X), h(f, g)\}$  is called an  $h$ -system on  $\mathfrak{A}$ <sup>(2)</sup>. Given two  $h$ -systems  $\{G_1(X), h(f, G_1)\}$ ,  $\{G_2(X), h(f, G_2)\}$  on  $\mathfrak{A}$  and a definite  $Y \in \mathfrak{A}$ , a homomorphism  $\eta: G_1(Y) \rightarrow G_2(Y)$  is natural if for each  $f: Y \rightarrow Y$  the diagram

$$\begin{array}{ccc} G_1(Y) & \xrightarrow{h(f, G_1)} & G_1(Y) \\ \eta \downarrow & & \downarrow \eta \\ G_2(Y) & \xrightarrow{h(f, G_2)} & G_2(Y) \end{array}$$

is commutative.

Restrict attention to one  $h$ -system on  $\mathfrak{A}$  and a definite  $f: X \rightarrow Y$ . If  $f$  is a homotopy equivalence [4; 1133], then  $h(f, G): G(X) \approx G(Y)$ . If  $f$  has only a left homotopy inverse<sup>(3)</sup>, then  $h(f, G)$  is an isomorphism of  $G(X)$  into  $G(Y)$  (and onto a direct summand if  $G(Y)$  is abelian). The object of this note is to remark that if one considers two  $h$ -systems on  $\mathfrak{A}$  something more may be said in this latter case, and to give a simple application.

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(1) In other terms,  $G$  is a functor defined on the category  $(X, \alpha)$  where  $X \in \mathfrak{A}$  and  $\alpha$  is a homotopy class of maps, with values in the category of groups.

(2) The concept simply lists the properties in the category  $\mathfrak{A}$  of say, the homotopy groups, or the singular homology groups, or the Čech groups based on all coverings, that are required in the simple proof of 1.1. Note that Čech groups based on finite coverings in non-compact spaces do not form an  $h$ -system.

(3) That is, there is a  $g: Y \rightarrow X$  with  $gf \approx$  identity.

In fact,

1.1 THEOREM. Let  $\{G_1(X), h(f, G_1)\}, \{G_2(X), h(f, G_2)\}$  be two  $h$ -systems on  $\mathfrak{A}$ . Assume that at  $Y \in \mathfrak{A}$  there is a natural isomorphism  $\eta: G_1(Y) \approx G_2(Y)$ . If for any  $X \in \mathfrak{A}$  there are maps

$$X \xrightarrow{f} Y \xrightarrow{g} X$$

with  $gf = \text{identity}$ , then  $G_1(X)$  is isomorphic to  $G_2(X)$ <sup>(4)</sup>.

PROOF. Setting  $X_i = G_i(X), Y_i = G_i(Y), f_i = h(f, G_i), g_i = h(g, G_i), i = 1, 2$ , one has the diagram

$$(1) \quad \begin{array}{ccccccc} X_1 & \xrightarrow{f_1} & Y_1 & \xrightarrow{g_1} & X_1 & \xrightarrow{f_1} & Y_1 \\ & & \downarrow \eta & & & & \downarrow \eta \\ X_2 & \xrightarrow{f_2} & Y_2 & \xrightarrow{g_2} & X_2 & \xrightarrow{f_2} & Y_2 \end{array}$$

Since the  $f_i$  are isomorphisms into, and the  $\eta$  is an isomorphism onto, it suffices to show  $\eta[\text{Image } f_1] = \text{Image } f_2$ . If  $y_1 \in \text{Image } f_1$  then  $y_1 = f_1(x_1)$  for some  $x_1 \in X_1$  so that  $\eta(y_1) = \eta f_1 g_1 f_1(x_1) = f_2 g_2 \eta f_1(x_1)$  showing  $\eta(y_1) \in \text{Image } f_2$ . Conversely, if  $\eta(y_1) \in \text{Image } f_2$ , then  $\eta(y_1) = f_2(x_2)$  for some  $x_2 \in X_2$  and  $\eta[f_1 g_1(y_1)] = f_2 g_2 \eta(y_1) = f_2 g_2 f_2(x_2) = \eta(y_1)$  so  $y_1 = f_1 g_1(y_1)$  and  $y_1 \in \text{Image } f_1$ .

1.2 REMARK. The isomorphism  $\tilde{\eta} = f_2^{-1} \eta f_1: X_1 \approx X_2$  in 1.1. makes the diagram (1) commutative. From this follows: if there is an  $\tilde{\eta}: G_1(X) \rightarrow G_2(X)$  which, for all  $\varphi: X \rightarrow Y$  makes

$$\begin{array}{ccc} X_1 & \xrightarrow{h(\varphi, G_1)} & Y_1 \\ \tilde{\eta} \downarrow & & \downarrow \eta \\ X_2 & \xrightarrow{h(\varphi, G_2)} & Y_2 \end{array}$$

commutative, then under the hypotheses of 1.1,  $\tilde{\eta}: X_1 \approx X_2$ .

2. Application. Recall [4; 1133] that a space  $P$  dominates a space  $X$  if and only if there are maps  $z: X \rightarrow P, g: P \rightarrow X$  with  $gz = \text{identity}$ .

(4) With no naturality condition on  $\eta: G_1(Y) \approx G_2(Y)$ , the case that  $f: X \rightarrow Y$  has a right homotopy inverse leads trivially to the result:  $G_1(X)/\text{kernel } h(f, G_1)$  is isomorphic to  $G_2(X)/\text{kernel } h(f, G_2)$ .

**2.1 THEOREM.** *In any space dominated by a CW polytope<sup>(5)</sup>, the CECH homology groups based on all coverings [1; 278-292] and the singular homology groups [3; 407-447] are isomorphic in each dimension.*

**PROOF.** Since both these theories, with the usual induced homomorphisms form an  $h$ -system on, say,  $\mathfrak{A} = \{X, P\}$  and since the two groups are isomorphic naturally in each dimension in polytopes, the result follows from 1.1.

**REMARK.** Any metric space that is an absolute neighborhood retract for metric spaces is dominated by a CW polytope. [2; 365].

### BIBLIOGRAPHY

- [1] C. H. DOWKER, *CECH cohomology theory and the axioms*, Ann. Math. **51** (1950), 278-292.
- [2] J. DUGUNDJI, *An extension of Tietze's theorem*, Pacific J. Math. **1** (1951), 353-367.
- [3] S. EILENBERG, *Singular homology theory*, Ann. Math. **45** (1944), 407-447.
- [4] J. H. C. WHITEHEAD, *On the homotopy type of ANR's*, Bull. Am. Math. Soc. **54** (1948), 1133-1145.
- [5] ———, *Simplicial spaces, nuclei, and m-groups*, Proc. London Math. Soc. **45** (1939), 243-327.

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(5) The CW topology on the polytope [5; 315-321] is taken for definiteness