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# A NOTE ON GAUSS' "SERIERUM SINGULARIUM" *

BY L. CARLITZ
*Duke University, Durham, North Carolina, U. S. A.*

**1.** Put

$$(a)_m = (1-a)(1-ax)\cdots(1-ax^{m-1}), \quad (x)_0 = 1,$$

and

$$\begin{bmatrix} m \\ r \end{bmatrix} = \frac{(x)_m}{(x)_r\,(x)_{m-r}}, \quad \begin{bmatrix} m \\ 0 \end{bmatrix} = \begin{bmatrix} m \\ m \end{bmatrix} = 1 .$$

In connection with the evaluation of the sum $\sum_{s=0}^{m-1} e^{2\pi i s^2/m}$, GAUSS [1] proved the identity

$$(1) \qquad \sum_{r=0}^{2m} (-1)^r \begin{bmatrix} 2m \\ r \end{bmatrix} = (1-x)(1-x^3)\cdots(1-x^{2m-1}).$$

The proof, which is reproduced in [3; CHAPTER 7] and [4; CHAPTER 5] is a verification and it is by no means evident what suggested the left member of (1). Indeed the form of the left member is rather surprising when compared with the identity

$$\sum_{r=0}^{m} (-1)^r \begin{bmatrix} m \\ r \end{bmatrix} x^{r(r-1)/2}\, t^r = (1-t)(1-xt)\cdots(1-x^{m-1}t).$$

We wish to point out in this note that (1) can be obtained in a more natural way using only EULER's identity [2, p. 278]

$$(2) \qquad \prod_0^{\infty} (1-x^m t)^{-1} = \sum_0^{\infty} t^m/(x)_m .$$

Indeed in (2) replace $t$ by $-t$, so that

$$(3) \qquad \prod_0^{\infty} (1+x^m t)^{-1} = \sum_0^{\infty} (-1)^m t^m/(x)_m .$$

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If we multiply corresponding members of (2) and (3) we find that the right side

$$= \sum_{m=0}^{\infty} \frac{t^m}{(x)_m} \sum_{r=0}^{m} (-1)^r \begin{bmatrix} m \\ r \end{bmatrix},$$

while the left side

$$= \prod_{0}^{\infty} (1 - x^{2m} t^2)^{-1} = \sum_{0}^{\infty} t^{2m}/(x^2)'_m,$$

where

$$(x^2)'_m = (1 - x^2)(1 - x^4) \cdots (1 - x^{2m})$$

Consequently

$$\sum_{r=0}^{2m} (-1)^r \begin{bmatrix} 2m \\ r \end{bmatrix} = \frac{(x)_{2m}}{(x^2)'_m} = (1 - x)(1 - x^3) \cdots (1 - x^{2m-1}),$$

which proves (1).

If in place of (2) we use the companion formula [2, p. 278]

$$\prod_{0}^{\infty} (1 + x^m t) = \sum_{0}^{\infty} x^{m(m-1)/2} t^m/(x)_m, \tag{4}$$

we get in exactly the same way

$$\sum_{0}^{2m} (-1)^r x^{(m-r)^2} \begin{bmatrix} 2m \\ r \end{bmatrix} = (-1)^m (1 - x)(1 - x^3) \cdots (1 - x^{2m-1}) \tag{5}$$

However (5) can be obtained from (1) by replacing $x$ by $x^{-1}$ and simplifying.

**2.** Gauss [1] also proved the formula

$$\sum_{r=0}^{m} x^{r/2} \begin{bmatrix} m \\ r \end{bmatrix} = \prod_{r=1}^{m} (1 + x^{r/2}). \tag{6}$$

This can also be obtained from (2). In (2) replace $t$ by $x^{1/2} t$ and multiply together corresponding members. We get

$$\sum_{m=0}^{\infty} \frac{t^m}{(x)_m} \sum_{r=0}^{m} x^{r/2} \begin{bmatrix} m \\ r \end{bmatrix} = \prod_{0}^{\infty} (1 - x^{m/2} t)^{-1},$$

so that

$$\sum_{r=0}^{m} x^{r/2} \begin{bmatrix} m \\ r \end{bmatrix} = \frac{(1-x)(1-x^2)\cdots(1-x^m)}{(1-x^{1/2})(1-x)\cdots(1-x^{m/2})}$$
$$= (1+x^{1/2})(1+x)\cdots(1+x^{m/2}),$$

which proves (6).

**3.** The formulas (1) and (6) can be generalized in the following way. In place of (2) we use the more general identity

$$\prod_{0}^{\infty} \frac{1-a x^m t}{1-x^m t} = \sum_{0}^{\infty} \frac{(a)_m}{(x)_m} t^m . \tag{7}$$

Then operating on (7) exactly as in the proof of (1), we get

$$\sum_{r=0}^{2m} (-1)^r \begin{bmatrix} 2m \\ r \end{bmatrix} (a)_r (a)_{2m-r} = \prod_{s=0}^{m-1} (1-x^{2s+1})(1-a^2 x^{2s}) . \tag{8}$$

Similarly, as in the proof of (6), we get

$$\sum_{r=0}^{m} x^{r/2} \begin{bmatrix} m \\ r \end{bmatrix} (a)_r (a)_{m-r} = \prod_{s=1}^{m} (1+x^{s/2})(1-a x^{(s-1)/2}) . \tag{9}$$

For $a=0$, (8) and (9) reduce to (1) and (6) respectively.
For $a=x^{1/2}$, they become

$$\sum_{r=0}^{2m} (-1)^r \begin{bmatrix} 2m \\ r \end{bmatrix} (x^{1/2})_r (x^{1/2})_{2m-r} = \prod_{0}^{m-1} (1-x^{2s+1})^2 , \tag{10}$$

$$\sum_{r=0}^{m} x^{1/2} \begin{bmatrix} m \\ r \end{bmatrix} (x^{1/2})_r (x^{1/2})_{2m-r} = (x)_m ; \tag{11}$$

the left members of (10) and (11) can be simplified.

If $n=2m+1$ is an arbitrary odd number and we take in (8) $x=e^{-4\pi i/n}$, we get after a little manipulation

$$S_n(a) = \sum_{r=0}^{2m} \varepsilon^{(m-r)^2} (a)_r (a)_{2m-r} = \prod_{0}^{m-1} 2i \sin \frac{2\pi(2s+1)}{n} \cdot \prod_{0}^{m-1} (1-a^2 \varepsilon^{-4s})$$

$$(\varepsilon = e^{2\pi i/n}).$$

Since the first product is $n^{1/2}$ or $in^{1/2}$ according as $n \equiv 1$ or $3 \pmod 4$, it follows that for $|a| < 1$ we have

$$\begin{aligned} &\mathrm{R}(\mathrm{S}_n(a)) > 0 \qquad (n \equiv 1 \pmod 4) \\ &\mathrm{I}(\mathrm{S}_n(a)) > 0 \qquad (n \equiv 3 \pmod 4); \end{aligned} \tag{12}$$

for $a = 0$, (12) reduces to the familiar result for Gaussian sums.

## REFERENCES

[1] C. F. Gauss, *Summatio quarumdam serierum singularium*, Werke, Göttingen, 1876, vol. 2, pp. 9 – 45.
[2] G. H. Hardy and E. M. Wright, *An introduction to the theory of numbers*, Oxford, 1938.
[3] G. B. Mathews, *Theory of numbers*, Cambridge, 1892.
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