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ON THE DIMENSION OF C(X) *

BY BYRON H. McCANDLESS

Rutgers — The State University — New Brunswick — U. S. A.

Let X be a T_1 space, and let C(X) be the family of all bounded, continuous, real-valued functions define on X. It is well known that C(X) is a BANACH space with norm $|f| = \sup_{x \in X} |f(x)|$. In many applications the linear dimension of C(X) (which we shall denote by lin dim C(X)) is of crucial importance, so it is desirable to be able to deduce lin dim C(X) from easily determined properties of the space X. In this note we shall show that if X consists of a finite number n of points, then lin dim $C(X) = n$, and if X is completely regular and has an infinite number of points, then lin dim $C(X) \geq c$, where c is the cardinality of the continuum.

First, suppose that X consists of a finite number of distinct points $x_1, \cdots, x_n$. Define n mappings $f_i : X \to R$ $(i = 1, \cdots, n)$ where R is the space of real numbers, by

$$f_i(x_i) = 1, \quad f_i(x_j) = 0 \text{ for } i \neq j.$$

These mappings are clearly continuous and bounded. To show that they are linearly independent, consider the relation

$$r_1 f_1 + \cdots + r_n f_n = 0,$$

where $r_i \in R$, $i = 1, \cdots, n$. Since

$$[r_1 f_1 + \cdots + r_n f_n](x_i) = r_i, \quad i = 1, \cdots, n,$$

it follows that $r_i = 0$, $i = 1, \cdots, n$ and therefore $f_1, \cdots, f_n$ are linearly independent. Now let $f : X \to R$ be an arbitrary mapping. Then clearly

$$f = f(x_1) f_1 + \cdots + f(x_n) f_n$$

* Received May, 1961.

and therefore the mappings $f_1, \cdots, f_n$ form a basis for $C(X)$. Hence lin dim $C(X) = n$ as stated above.

The second part of the result mentioned above will follow from the following lemma:

LEMMA. *Let* X *be a regular* T_1 *space which contains an infinite number of points. Then there exists in* X *an infinite collection* $\{U_i\}$ *of non-empty open sets such that* $U_i \cap U_j = \phi$ *for* $i \neq j$.

PROOF. We shall construct the collection $\{U_i\}$ by induction. Define $X_0 = X$ and let x, x' be distinct points of X_0. Since X_0 is regular there exist neighborhoods U_0, U_0' of x, x' respectively such that $\overline{U}_0 \cap \overline{U}_0' = \phi$. Therefore $X_0 = (X_0 - \overline{U}_0) \cup (X_0 - \overline{U}_0')$ so we may assume that $X_0 - \overline{U}_0$ contains an infinite number of points. Define $X_1 = X_0 - \overline{U}_0$. X_1 is open in X_0 and contains an infinite number of points. Also, X_1 is regular T_1 since regularity and T_1 are hereditary properties. Therefore the above argument can be repeated to yield a non-empty set U_1 which is open in X_1 (and hence in X_0) and such that $X_1 - \overline{U}_1$ contains an infinite number of points (closure being taken in X_1). Suppose that we have obtained spaces $X_0, X_1, \cdots, X_{n-1}$ and disjoint non-empty open subsets $U_0, U_1, \cdots, U_{n-1}$ of $X_0, X_1, \cdots, X_{n-1}$ respectively satisfying (*a*) X_i is open in X_{i-1}, $i = 1, \cdots, n-1$ and (*b*) $X_i - \overline{U}_i$ contains an infinite number of points, $i = 0, 1, \cdots, n-1$, where closure is taken in X_i. Define $X_n = X_{n-1} - \overline{U}_{n-1}$. X_n is open in X_{n-1} (and hence in X_0), is regular T_1, and contains an infinite number of points. Therefore the same argument used above yields a non-empty open set U_n of X_n (and hence of X_0) such that $X_n - \overline{U}_n$ contains an infinite number of points. Clearly the sets $U_1, \cdots, U_n$ are disjoint non-empty open sets of X_0. This completes the induction and finishes the proof of the lemma.

Now let X be a completely regular T_1 space with an an infinite number of points and let $\{U_i\}_{i=1}^{\infty}$ be the infinite collection of disjoint non-empty open sets of X provided by lemma. Select a point $x_i \in U_i, i = 1, 2, 3, \cdots$. By complete regularity there exists for each i a bounded continuous mapping $f_i: X \to R$ such that $f_i(x_i) = 1$, $f_i(X - U_i) = 0$ (The usual definition of complete regularity gives a mapping $g_i: X \to R$ with $g_i(x_i) = 0$, $g_i(X - U_i) = 1$. We need only set $f_i = 1 - g_i$ for each i). Then the collection $\{f_i\}$ is linearly independent. For consider the relation

$$r_1 f_{i_1} + \cdots + r_n f_{i_n} = 0$$

where $i_1, \cdots, i_n$ are distinct positive integers. Since

$$[r_1 f_{i_1} + \cdots + r_n f_{i_n}](x_{i_j}) = r_j$$

it is clear that $r_j = 0, j = 1, \cdots, n$. Therefore the collection $\{f_i\}$ is linearly independent as asserted, and shows that lin dim C(X) is infinite. It now follows from [1] that lin dim $C(X) \geqq c$, where c is the cardinality of the continuum. If C(X) is separable, then it also follows from [1] that lin dim $C(X) = c$. This is the case, for example, when X is separable metric ([2, p. 186]).

BIBLIOGRAPHY

[1] A. E. TAYLOR, *Introduction to Functional Analysis,* Wiley, 1958 p. 191.
[2] M. WOJDYSLAWSKI, *Rétractes absolus et hyperespaces des continus,* Fund. Math. **32** (1939) pp. 184-192.