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AN EXTENSION OF A THEOREM OF NIELSEN *

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If p is a prime > 3, $1 < r < \frac{1}{2}(p-1)$, GLAISHER [3] proved

$$(1.1)\qquad \begin{aligned} A_{2r}^{(p)} &\equiv -\frac{1}{2r}\, p\, B_{2r} \pmod{p^2}, \\ A_{2r+1}^{(p)} &\equiv \frac{2r+1}{4r}\, p^2\, B_{2r} \pmod{p^3}, \end{aligned}$$

where B_m denotes the m-th BERNOULLI number in the notation of NÖRLUND and $A_m^{(p)}$ is defined by

$$(x+1)(x+2)\cdots(x+p-1) = \sum_{s=0}^{p-1} A_s^{(p)}\, x^{p-1-s}.$$

CARLITZ [1] extended GLAISHER's result to show that for $1 < r < \frac{1}{2}(p-1)$

$$(1.2)\qquad \begin{aligned} A_{2r}^{(p)} &\equiv -\frac{1}{2r}\, p\, B_{2r} + \frac{1}{r}\, p^2 \sum_{s=1}^{r-1} \frac{1}{4s} B_{2s} B_{2r-2s} \pmod{p^3}, \\ A_{2r+1}^{(p)} &\equiv -\frac{1}{4r}\, p^2 (p-2r-1)\, B_{2r} - \frac{2r+1}{2r}\, p^3 \sum_{s=1}^{r-1} \frac{1}{4s} B_{2s} B_{2r-2s} \pmod{p^4}. \end{aligned}$$

In this note we develop congruences for the numbers $\overline{A}_r^{(p)}$ defined by

$$\overline{A}_r^{(p)} = \frac{1}{p!} \sum_{s=0}^{p} (-1)^{p-s} \binom{p}{s} s^{p+r}.$$

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* Received February 1964.

Since for $1 < r < \frac{1}{2}(p-1)$ NIELSEN [4, p. 338] proved

(1.3)
$$\overline{A}_{2r}^{(p)} \equiv \frac{1}{2r} p B_{2r} \pmod{p^2},$$
$$\overline{A}_{2r+1}^{(p)} \equiv \frac{2r+1}{4r} p^2 B_{2r} \pmod{p^3},$$

it seems natural to seek congruences analogous to (1.2). Also, congruences for the $A_r^{(p)}$ and $\overline{A}_r^{(p)}$ numbers yield congruences for the BERNOULLI numbers of higher order since

$$A_m^{(p)} = (-1)^m \binom{p-1}{m} B_m^{(p)}$$

and

$$\overline{A}_m^{(p)} = \binom{p+m}{p} B_m^{(-p)}.$$

Let

(1.4)
$$S_m = S_m(p) = 1^m + 2^m + \cdots + p^m.$$

NIELSEN [4, p. 337] has proved

(1.5)
$$n A_n^{(p)} = \sum_{s=0}^{n-1} S_{n-s} \overline{A}_s^{(p)}.$$

Hence

(1.6)
$$(2r+1)\overline{A}_{2r+1}^{(p)} - S_1 \overline{A}_{2r}^{(p)} = S_{2r+1} + \overline{A}_1^{(p)} S_{2r} + \sum_{s=1}^{r-1} \overline{A}_{2s}^{(p)} S_{2r+1-2s} + \sum_{s=1}^{r-1} \overline{A}_{2s+1}^{(p)} S_{2r-2s}.$$

Further [4, p. 147]

(1.7)
$$S_m = p^m + \frac{1}{m+1} \sum_{s=1}^{m+1} \binom{m+1}{s} B_{m+1-s} p^s$$

which implies for $1 < r < \frac{1}{2}(p-1)$, since $B_{2r+1} = 0$,

(1.8)
$$S_{2r} \equiv p B_{2r} \pmod{p^3}$$
$$S_{2r+1} \equiv \frac{2r+1}{2r} p^2 B_{2r} \pmod{p^4}.$$

By means of (1.8) and (1.3) the sums on the right of (1.6) may be written

(1.9)
$$\sum_{s=1}^{r-1} \overline{A}_{2s}^{(p)} S_{2r+1-2s} + \sum_{s=1}^{r-1} \overline{A}_{2s+1}^{(p)} S_{2r-2s}$$
$$\equiv (2r+2)\, p^3 \sum_{s=1}^{r-1} \frac{1}{4s} B_{2s} B_{2r-2s} \pmod{p^4}.$$

Recalling that $\overline{A}_1^{(p)} = \frac{1}{2} p(p+1)$ we find from (1.7) that

(1.10)
$$S_{2r+1} + \overline{A}_1^{(p)} S_{2r} \equiv \tfrac{1}{2} p^2 (p + 2r + 2) B_{2r} \pmod{p^4}.$$

Combining (1.6), (1.9) and (1.10) we obtain

(1.11)
$$(2r+1)\overline{A}_{2r+1}^{(p)} - \tfrac{1}{2} p(p+1)\overline{A}_{2r}^{(p)}$$
$$\equiv \tfrac{1}{2} p^2 (p+2r+2) B_{2r} + (2r+2) p^3 \sum_{s=1}^{r-1} \frac{1}{4s} B_{2s} B_{2r-2s} \pmod{p^4}.$$

If we now compare [2, p. 246]

(1.12)
$$\tfrac{1}{2} p(p + 2r + 1)\overline{A}_{2r}^{(p)} - \overline{A}_{2r+1}^{(p)} \equiv 0 \pmod{p^4}$$

with (1.11) we have

(1.13)
$$2r\,\overline{A}_{2r+1}^{(p)} \equiv \tfrac{1}{2} p^2 (p + 2r + 1) B_{2r} + (2r+1) p^3 \sum_{s=1}^{r-1} \frac{1}{4s} B_{2s} B_{2r-2s} \pmod{p^4}.$$

Elimination of $\overline{A}_{2r+1}^{(p)}$ between (1.12) and (1.13) gives the residue of $\overline{A}_{2r}^{(p)} \pmod{p^3}$. Hence we have proved the following theorem, which is similar to (1.2):

THEOREM 1. *For* $1 < r < \frac{1}{2} p(p-1)$, p *a prime* > 3,

(1.14)
$$\overline{A}_{2r}^{(p)} \equiv \frac{1}{2r} p B_{2r} + \frac{1}{r} p^2 \sum_{s=1}^{r-1} \frac{1}{4s} B_{2s} B_{2r-2s} \quad (\textit{mod}\ p^3),$$
$$\overline{A}_{2r+1}^{(p)} \equiv \frac{1}{4r} p^2 (p+2r+1) B_{2r} + \frac{2r+1}{2r} p^3 \sum_{s=1}^{r-1} \frac{1}{4s} B_{2r-2s} \quad (\textit{mod}\ p^4).$$

Moreover, (1.2) and (1.14) yield

THEOREM 2. *For* $1 < r < \frac{1}{2}(p-1)$, p *a prime* > 3

$$\bar{A}_{2r}^{(p)} - A_{2r}^{(p)} \equiv \frac{p}{r} B_{2r} \pmod{p^3},$$

$$\bar{A}_{2r+1}^{(p)} + A_{2r+1}^{(p)} \equiv \frac{2r+1}{2r} p^2 B_{2r} \pmod{p^4}.$$

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