PORTUGALIAE MATHEMATICA

VOLUME 25

1966

Edição de

«GAZETA DE MATEMÁTICA, LDA.»

PORTUGALIAE MATHEMATICA
Rua Nova da Trindade, 1, 5.º-S
LISBOA-2 (PORTUGAL)

HERMANN & C.ie, Editeurs
6, Rue de la Sorbonne
PARIS (5eme)

PORTUGALIAE MATHEMATICA
Vol. 25 — Fas. 3 — 1966

EXPANSIONS OF SOLUTIONS TO A HYPOELLIPTIC DIFFERENTIAL EQUATION IN TERMS OF POLYNOMIALLIKE SOLUTIONS *

BY H. J. BIESTERFELDT, Jr.
University of Wisconsin, Milwaukee, Wisconsin — U. S. A.

This paper concerns the expansion of solutions to a hypoelliptic equation with constant coefficients [1]. In one special case our main theorem concerns homogeneous polynomial expansions, and in the general case there is no essential change in result. We give the exact domain of convergence for the series considered.

We wish to examine expansions valid in the real (x, y) plane for solutions to the equation:

$$(\sigma) \qquad [D_x^2 + i D_y^2 - \alpha] \, \sigma(x, y) = 0$$

where $\alpha = \beta(1 + i)$ and β is real. The condition on α implies that all solutions $\sigma(x, y)$ are complex valued analytic functions of the real variables x and y. This result is seen since from the criteria given by HORMANDER ([4] p. 232), all solutions to (σ) are infinitely differentiable. Moreover, we may separate (σ) into two equations satisfied by the real and the imaginary parts of $\sigma(x, y)$. Since these parts are infinitely differentiable, we may find an elliptic equation of order 4 which is satisfied by both the real and the imaginary parts. This equation is derived much as LAPLACE's equation is obtained from the CAUCHY-RIEMANN equations. Now all solutions to this elliptic equation are analytic, so $\sigma(x, y)$ itself must be analytic. For this reason we may deal with only solutions which have TAYLOR Series, and without loss of generality we will assume that the origin is a regular point, and use MACLAURIN Series.

---

* Received May, 1964.

In view of the above, we make the change of variables $z = x + y\sqrt{i}$, $z^* = x - y\sqrt{i}$. Writing $\sigma^*(z, z^*) = \sigma(x, y)$ where the variables are related as above, we obtain the equation

$$(\sigma^*) \qquad [D_z D_{z^*} - \alpha'] \sigma^*(z, z^*) = 0$$

where $\alpha' = \alpha/4$. Direct computation will verify

PROPOSITION 1. *Formally, the function*

$$(1) \qquad f(z, z^*) = \sum_{n=0}^{\infty} [A_n z^n + B_n z^{*n}] \cdot J_n(2\sqrt{(\alpha' z z^*)})/(\alpha' z z^*)^{n/2}$$

is a solution to (σ^*), *where* J_n *is the* BESSEL *function of the first kind and* A_n *and* B_n *are constants.*

Let $\varphi_m(x, y)$ be given by

$$\varphi_m(x, y) = J_m(2\sqrt{(\alpha'(x+y\sqrt{i})(x-y\sqrt{i}))})/(\alpha'(x+y\sqrt{i})(x-y)\sqrt{i}))^{m/2}.$$

Then easy computation will verify (see [2]).

PROPOSITION 2. *For* (x, y) *in any bounded domain, we have uniformly in* (x, y) *as* $m \to \infty$,

$$\varphi_m(x, y) = (1 + O(1/m))/m!.$$

If $\alpha = 0$, then series (1) is a series of homogeneous polynomials in z and z^*. For equations with real coefficients this has been examined by WIDDER [3].

We write:

$$f(z) = \sum_{m=0}^{\infty} a_m z^m / m!$$

$$(3) \qquad g(z^*) = \sum_{n=0}^{\infty} b_m z^{*m} / m!, \quad a_0 = b_0.$$

PROPOSITION 3. *Let* $f(z)$ *and* $g(z^*)$ *be given functions of* z *and* z^* *respectively, both analytic near zero. Then there is a unique* *solution* $\sigma^*(z, z^*)$ *to equation* (σ^*), *analytic in* z *and* z^*, *for which* $\sigma^*(z, 0) = f(z)$ *and* $\sigma^*(0, z^*) = g(z^*)$. (*We assume* $f(0) = g(0)$).

PROOF. In the formal solution given by Proposition 1, let $A_m = a_m$ and $B_m = b_m$. Let K be any compact set in the domain of convergence of series (2). Using the estimate given in Proposition 2, the series

$$\sum A_m z^m J_m(2\sqrt{(\alpha' z z^*)})/(\alpha' z z^*)^{m/2}$$

converges absolutely for $z \in K$ and z^* restricted to all bounded set real. Thus the series is a solution to (σ^*), and for $z^* = 0$, it takes the value $f(z)$. For $z = 0$, it takes the value 0. The similar series using B_m and z^{*m} converges to a solution for z^* inside the circle of convergence of series (3), for the same reasons. Thus for z inside the circle of convergence of (2) and z^* inside the circle of convergence of (3), series (1) with $A_m = a_m$ and $B_m = b_m$ (except $B_0 = 0$) is a solution to (σ^*) and satisfies the given conditions when z or z^* is zero.

If we assume a power series solution in powers of z and z^*, for (σ^*), ond substitute into the equation, we obtain a recurrence relation which determines the coefficient of $z^m z^{*n}$ in terms of that of $z^{m-1} z^{*n-1}$ only. Hence it is clear that there is at most one analytic solution, and we already know that all solutions are analytic.

Hereafter $\sigma(x, y)$ denotes the function

$$(4) \qquad \sigma(x, y) = \sum_{m=0}^{\infty} [A_m (x + y\sqrt{i})^m + B_m (x - y\sqrt{i})^m] \varphi_m(x, y)$$

which we have just shown to be a solution to (σ).

Using the usual expression for the radius of convergence for series (2) and (3), and recalling that if the «A_m» portion of (4) converges and the «B_m» portion diverges, then the sum diverges, we have

PROPOSITION 4. *Let* r_1 *and* r_2 *be given by*

$$1/r_1 = \text{Lim Sup} \, |A_m/m!|^{1/m}$$
$$1/r_2 = \text{Lim Sup} \, |B_m/m!|^{1/m}.$$

Then series (4) converges absolutely if simultaneously

$$|x+y\sqrt{i}|<r_1 \quad and \quad |x-y\sqrt{i}|<r_2.$$

Series (4) diverges if simultaneously

$$|x+y\sqrt{i}|>r_1 \quad and \quad |x-y\sqrt{i}|<r_2$$

as well as if simultaneously

$$|x+y\sqrt{i}|<r_1 \quad and \quad |x-y\sqrt{i}|>r_2.$$

We sketch the domains represented in the above proposition, for x and y real. We take r_1 and r_2 to be finite, nonzero, and unequal. This yields a pair of ellipses which are never circles, and with axes inclined by $\pm 45°$. By $\mathcal{C}$ is meant the region interior to both, and by $\mathcal{D}$, the region interior to one, and exterior to the other. [Figure 1]. Then $\mathcal{C}$ is the domain for which Propo-

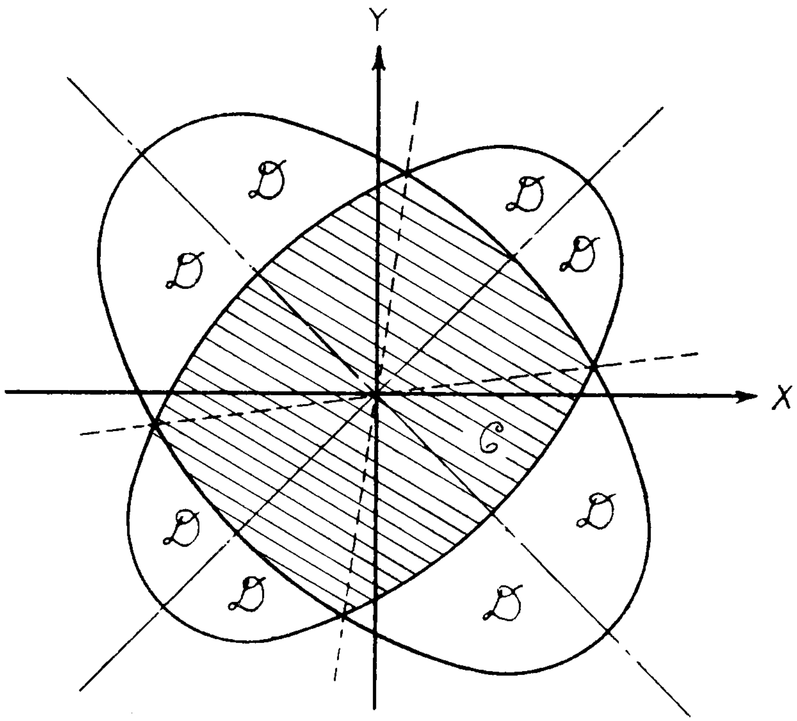

Figure 1

sition 4 assures convergence, and $\mathfrak{D}$ is that for which it assures divergence.

By analogy with WIDDER we expect that we cannot have convergence in a set of positive measure outside $\mathfrak{C}$. We prove a stronger result.

THEOREM 1. *The series (4) converges absolutely in* $\mathfrak{C}$, *and uniformly on compact subsets of* $\mathfrak{C}$. *It may converge on the boundary of* $\mathfrak{C}$. *Convergence may also take place on intervals, closed or not symmetric to the origin, lying on rays through the intersection of the ellipses. The series diverges at all other points.*

PROOF. Suppose that the series (4) converges at the point (x_0, y_0) on the real x, y plane. That is, the following series converges for $t = 1$:

$$\sum [A_m (x_0 + y_0 \sqrt{i})^m + B_m (x_0 - y_0 \sqrt{i})^m] t^m \varphi_m (x, y).$$

Therefore it is absolutly convergent for $|t| < 1$ and uniformly convergent on closed subsets thereof. Thus for $|t| < 1$, the series

$$\sum [A_m (x_0 + y_0 \sqrt{i})^m + B_m (x_0 - y_0 \sqrt{i})^m] t^m / m!$$

by use of Proposition 2.

The estimate of Proposition 2 is uniform, so taking $x = t x_0$ and $y = t y_0$, $|t| \leqq 1$, we have

$$|\varphi_m (x, y)| \leqq B$$

for some $B < \infty$. Then

$$|A_m (x_0 + y_0 \sqrt{i})^m + B_m (x_0 - y_0 \sqrt{i})^m| \, |\varphi_m (t x_0, t y_0)| \, |t|^m \leqq$$
$$\leqq B \; |A_{m,0} (x_0 + y_0 \sqrt{i})^m + B_m (x_0 - y_0 \sqrt{i})^m| \cdot |t|^m / m!$$

Hence by comparison we have absolute and uniform convergence for $|t| \leqq r < 1$, for the series

$$\sum [A_m (t x^0 + t y_0 \sqrt{i})^m + B_m (t x_0 - t y_0 \sqrt{i})^m] \varphi_m (t x_0, t y_0).$$

Hence for $x = t x_0$ and $y = t y_0$, $|t| < 1$, we have convergence for the series (4). For t real this is just the symmetric ray from the origin to x_0, y_0, exclusive of the endpoints.

If the ellipses do not intersect, and one is finite, then one is entirely within the other. Any ray from outside both, to the origin, passes through $\mathcal{D}$, and hence using what we just established, the series cannot converge outside of both ellipses. The ellipses are never coincident, so if they intersect they do so in two pairs of points on rays from the origin. If a point outside both ellipses is not on one of these rays, convergence cannot occur since again this implies convergence for all points nearer the origin on a straight line. Hence the diagonals of $\mathcal{C}$ are the only possible lines extending outside outside where convergence can occur.

q. e. d.

We cannot get a stronger result since rays of convergence do occur. For example take

$$A_m = (m-1)!(1+\sqrt{1})^{-m}, B_m = -(m-1)!(1-\sqrt{i})^{-m}.$$

Then $r_1 = |1+\sqrt{i}|^{-1}$ and $r_2 = |1-\sqrt{i}|^{-1}$. The series last (3) converges for all $x = y$.

Choosing the coefficients as above except for the case m odd where they satisfy

$$|(m-1)! + B_m(1-\sqrt{i})^m| = m!\, r^{-m},$$

where $r > 2$, and also so that

$$|B_m| \leq (m-1)!\, |1-\sqrt{i}|^{-m},$$

we see that for $x = y$, (3) converges for $|x| < r$ and diverges for $|x| > r$. One easily checks that the above conditions can be satisfied.

REFERENCES

[1] L. SCHARTZ, *Théorie Des Distributions,* Hermann, Paris, 1957.
[2] G. N. WATSON, *Theory of Bessel Functions.*
[3] D. V. WIDDER, *Expansions in Series of Homogeneous Polynomial Solutions of the General Two-dtmensional Linear Partial Differential Equation of the Second Order with Constant Coefficients,* Duke J., **26**, pp. 599-604.
[4] L. HORMANDER, *On the Theory of General Partial Differential Operators,* Acta Mathematica **94** (1955).