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A SUFFICIENT CONDITION FOR QUASI-METRIZABILITY OF A TOPOLOGICAL SPACE*

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The quasi-metrization problem is analogous to the metrization problem. Therefore a topological space is quasi-metrizable if there exists a quasi-metric for which the quasi-metric topology is identical to the original topology. The definitions and theorems needed to establish the quasi-metric topology will be stated after which a sufficient condition for quasi-metrizability will be introduced. An example will then be offered to show that this is not a necessary condition.

DEFINITION 1. A quasi-metric for a set X is a function d on $X \times X$ to the non-negative reals such that for all x, y and z the following conditions are satisfied.

(1) $d(x,y)=0$ if and only if $x=y$

(2) $d(x,y)+d(y,z) \geqq d(x,z)$.

DEFINITION 2. A u-sphere or l-sphere of center b and radius $r>0$ is the set of points $\{x;\ d(b,x)<r,\ r>0\}$ or $\{x;\ d(x,b)<r,\ r>0\}$ respectively.

DEFINITION 3. A sequence $\{x_n\}$ is said to have the point b as a u-limit or l-limit if for each $r>0$, there exists a positive integer m such that for each $n>m, d(b,x_n)<r$ or $d(x_n,b)<r$ respectively.

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THEOREM 1. *Let* X *be a quasi-metric space. The family of all* u-*spheres for all* $x \in X$ *or the family of all l-spheres for all* $x \in X$ *is a base for a topology for* X.

The proof of this theorem is omitted. The proof is virtually the same as the corresponding proof for metric spaces. The topology determined by a base consisting of u-spheres or l-spheres will be called the quasi-metric topology. This topology is T_1. That is for any two points x and y there is an open set containing x but not y and an open set containing y but not x.

DEFINITION 4. A family $\mathcal{A}$ of subsets of a topological space is point finite if each point x of X is contained in at most a finite number of members of $\mathcal{A}$.

DEFINITION 5. A family $\mathcal{A}$ is σ-point finite if it is the union of a denumerable number of point finite subfamilies.

THEOREM 2. *Every topological space which is* T_1 *and has a* σ-*point finite base is quasi-metrizable.*

PROOF. Suppose $\mathcal{T}$ is T_1 and has a σ-point finite base denoted by $\mathcal{A}$. The family $\mathcal{A}$ is formed by taking the union of the point finite families $\mathcal{A}_1, \mathcal{A}_2, \cdots, \mathcal{A}_n, \cdots$ and the elements of each $\mathcal{A}_i$ are members of the base $\mathcal{B}$ for $\mathcal{T}$. Let x and y be arbitrary points of X. The function f_i will be constructed as follows.

(1) $f_i(x,y)=1$ if $\mathcal{A}_i$ contains at least one member of the base $\mathcal{B}$ for $\mathcal{T}$ which contains the point x but not the point y.

(2) $f_i(x,y)=0$ otherwise.

Then
$$d(x,y)=\sum_{i=1}^{\infty}\frac{f_i(x,y)}{2^i}.$$

It should be obvious that if $x=y$, then $f_i(x,y)=0$ for each $\mathcal{A}_i$. Suppose $x \neq y$. There must exist at least one $\mathcal{A}_i$ such that x is contained in a member B of the base $\mathcal{B}$ and $\mathcal{B} \in \mathcal{A}_i$. If $f_i(x,y)=0$ for all i, this would imply that every member of the base $\mathcal{B}$ which contained x also contained y. Since the topology is T_1, this is impossible. Therefore for at least one $\mathcal{C}_i$, $f_i(x,y)=1$ and hence if $x \neq y$, $d(x,y) \neq 0$.

Consider the values of $f_i(x,y)$, $f_i(y,z)$ and $f_i(x,z)$ for arbitrary x, y and z. Suppose $f_i(x,z)=1$. This implies that for the family $\mathcal{C}_i$, at least one member of the base $\mathcal{B}$ in $\mathcal{C}_i$ contains x but not z. Suppose $f_i(y,z)=0$. Then either y is not covered by $\mathcal{C}_i$ or every member of the base contained in $\mathcal{C}_i$ which contains y also contains z. If y is not covered by $\mathcal{C}_i$, then certainly $f_i(x,y)=1$ since x is covered by $\mathcal{C}_i$. Finally assume that every member of the base in $\mathcal{C}_i$ which contains y also contains z. Since x is covered by $\mathcal{C}_i$, if $f_i(x,y)=0$ then every member of the base in $\mathcal{C}_i$ which contains x also contains z. Thus $f_i(x,z)=0$ and we have a contradiction. It follows that $d(x,y)+d(y,z) \geq d(x,z)$.

Now let us assume that b is the topological limit of the sequence $\{x_n\}$. This says that a member B of $\mathcal{B}$ containing b contains the points x_n for all n greater than some positive integer m. Consider an arbitrary $\mathcal{C}_i$. If b is not covered by $\mathcal{C}_i$, then $f_i(b,x_n)=0$ for all n. If b is covered by $\mathcal{C}_i$, $f_i(b,x_n)=0$ for all n greater than some positive integer m_i. This is because the elements of $\mathcal{C}_i$ are members of the base $\mathcal{B}$ and only a finite number of such members contain the point b. Let r be greater than 0 and choose k so that $\frac{1}{2^k} < r$. Then for all $i \leq k$, find the maximum of the integers m_i for which $f_i(b,x_n)=0$ for all $n > m_i$. Call this maximum m'. Then

$$\sum_{i=1}^{\infty} \frac{f_i(b,x_n)}{2^i} \leq \sum_{k+1}^{\infty} \frac{1}{2^i} = \frac{1}{2^k} < r \quad \text{for all} \quad n > m'.$$

Hence for any $r > 0$ there exists an m' such that $d(b,x_n) < r$ for all $n > m'$.

Conversely suppose b is the u-limit of the sequence $\{x_n\}$ under the quasi-metric topology. Let B be any member of $\mathcal{B}$ which contains b. Now for some $\mathcal{C}_i$, $B \in \mathcal{C}_i$. Then if y belongs to the complement of B, $d(b,y) \geq \frac{1}{2^i}$ as $f_i(b,y)=1$ for all such y. The set $\left\{x;\ d(b,x) < \frac{1}{2^i}\right\}$ is therefore a subset of B and b is therefore the topological limit of the sequence $\{x_n\}$.

Example 1

Consider the closed interval [0,3]. Let the quasi-metric function d be defined as follows.

(1) $d(x,y)=0$ if $x=y$.

(2) $d(x,y)=\frac{1}{n}$ if x is an irrational number belonging to $[0,1]$ and $2+\frac{1}{n}<y\leqq 2+\frac{1}{n-1}$ $n=2,3,4,\cdots$.

(3) $d(x,y)=1$ otherwise.

Let I denote the set of all irrational numbers in the interval $[0,1]$. Consider the values of $d(x,y)$, $d(y,z)$ and $d(x,z)$. Suppose $d(x,z)=1$. Then $x\notin \mathrm{I}$ or $z\notin(2,3]$. If $x\notin \mathrm{I}$, then $d(x,y)=0$ or 1. If $d(x,y)=0$ then $x=y$ and $d(y,z)=d(x,z)$. If $z\notin(2,3]$, then $d(y,z)=0$ or 1. If $d(y,z)=0$, $y=z$ and $d(x,y)=d(x,z)$.

Once again consider the situation when $d(x,z)=\frac{1}{n_1}$. Then $x\in \mathrm{I}$ and $z\in(2,3]$. Now $d(y,z)=1$ or $\frac{1}{n_1}$ or 0. If $d(y,z)=0$, $y=z$ and $d(x,y)=d(x,z)$. Therefore for any x, y and z of X, $d(x,y)+d(y,z)\geqq d(x,z)$.

The members of the base for the quasi-metric topology will consist of u-spheres. That is the family of all sets of the form $\{z;\ d(x,z)<r, r>0\}$ for all $x\in \mathrm{X}$ is the base for the quasi-metric topology. We can make the following statements concerning members of the base for points belonging to I. If $x\in \mathrm{I}$, for any $r\leqq 1$, the u-sphere $\{y;\ d(x,y)<r\}$ contains no points of I other than x itself. Also if x and y are any two points of I, the u-spheres $\{t;\ d(x,t)<r_1, r_1>0\}$ and $\{t;\ d(y,t)<r_2, r_2>0\}$ have a non-null intersection. Consider a denumerable collection of families $\mathcal{A}_1,\mathcal{A}_2,\cdots,\mathcal{A}_n\cdots$ and suppose the members of each family $\mathcal{A}_i$ are members of the base for the quasi-metric topology. The collection of all of the families $\mathcal{A}_1,\mathcal{A}_2,\cdots,\mathcal{A}_n,\cdots$ will comprise the complete base for the topology only if at least one $\mathcal{A}_j$ contains a non-denumerable number of members of the base for points of I. This in turn implies that for some positive integer k, a non-denumerable number of members of $\mathcal{C}_j$ have radius $\geqq \frac{1}{k}$. It follows then that every point in the interval $\left(2, 2+\frac{1}{k}\right)$ is contained in a non-denumerable number of members of $\mathcal{C}_j$.

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