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CATEGORICAL INVESTIGATION OF Γ-GRADED Λ-ALGEBRAS (1)

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1. Introduction.

The aim of this article is to investigate the category of graded algebras. The notion of internally graded algebras is due to Chevally [1], [2] and that of externally graded algebras is due to MacLane and others [7]. We shall exhibit here the existence of direct and free product in the category of internally graded algebras and also give a simple alternative construction for free graded algebras than that given by MacLane. In § 6, we further characterize the varieties in this category.

In the sequel Λ, will denote a commutative ring with an identity. By a Λ-module A, we mean a left module with 1 as the unitary operator i. e. $1 \cdot u = u$ for all $u \in A$.

By an algebra over Λ, we shall mean an associative Λ-algebra without an identity. The free algebra has to be understood in this sense i. e. an algebra «without identity».

By a graded Λ-module A with Γ as its set of degrees we mean an object consisting of a Λ-module A, together with a direct sum decomposition $A = \Sigma_{\gamma \in \Gamma} A_\gamma$ of A into a family of submodules $(A_\gamma)_{\gamma \in \Gamma}$ indexed by the set Γ. If $x \in A$, belongs to A_γ for some γ, then x is called homogeneous of degree γ, clearly if $x \neq 0$, then there cannot exist more than one index γ, such that $x \in A_\gamma$.

Let M and N be graded modules over a ring Λ, with the same set of degrees Γ which *from now on we will assume to be a commutative additive group*. Let f be a linear mapping of $M \to N$.

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Let us denote by M_γ, N_γ the set of homogeneous elements of degree $\gamma (\gamma \in \Gamma)$ of M and N respectively. Then for any $\sigma \in \Gamma$, we shall say that f is homogeneous of degree σ, if $f(M_\gamma) \subset N_{\gamma+\sigma}$ for every $\gamma \in \Gamma$.

Obviously if $f: M \to N$ is homogeneous of degree σ and $g: N \to P$ is homogeneous of degree τ, then $fg: M \to P$ is homogeneous of degree $\sigma + \tau$.

A submodule N of M is called *homogeneous* or *graded submodule* if it satisfies the following condition.

If $x \in N$, then the homogeneous components of x are in N i. e. $N = \Sigma_{\gamma \in \Gamma} (M_\gamma \cap N)$. Clearly this decomposition defines on N, the structure of graded module with Γ as its set of degrees.

Definition 1.1. By a *Γ-graded Λ-algebra* we shall mean a Λ-algebra A together with a gradation of the underlying module having the group Γ as its set of degrees such that when x and x' are homogeneous elements of degree γ and γ' respectively, then xx' is homogeneous of ddgree $\gamma + \gamma'$, i. e. $A_\gamma \cdot A_{\gamma'} \subseteq A_{\gamma+\gamma'}$.

It is easy to see that if the graded algebra A has an idempotent $e \neq 0$, then e is homogeneous of degree 0 (0 is the identity of Γ).

First if e is an idempotent, then it must belong to a homogeneous component, otherwise if $e = \Sigma_{\gamma \in \Gamma} e_\gamma$, then $e = ee$ will lead to two different representations of e which is impossible. Thus if $e \in A_\gamma$; then $e \cdot e \in A_{2\gamma}$ i. e. $e \in A_{2\gamma}$ and therefore $\gamma = 0$ i. e. $e \in A_0$.

Definition 1.2. Let A and B be two graded Λ-algebras having the same set of degree Γ, then $f: A \to B$ is called a *homomorphism* of graded algebras, if

1) f is a linear mapping of degree zero of the module A into the module B.

2) $f(xy) = f(x)f(y)$ for $x, y \in A$.

Notion of monomorphisms, epimorphisms, isomorphisms etc. are in the usual sense.

The graded algebras and their homomorphisms form a category which we shall denote by $\mathcal{G}_\Lambda^\Gamma$. (The category of Γ graded Λ-algebras).

DEFINITION 1. 3. If A is a graded algebra and B a subalgebra of A, then we call B a *graded subalgebra* if it is a graded submodule of A. It is clear that B has become a graded algebra when equipped with the gradation of A.

Immediately one notes that if A and B are two Γ-graded Λ-algebra and $f: A \to B$ be a homomorphism of graded algebras, then $f(A)$ is a graded subalgebra of B.

DEFINITION 1. 4. By a *graded ideal* J of a graded algebra A we shall understand that J is an ideal of the graded algebra A (two sided) and furthermore J is a graded submodule of A.

PROPOSITION 1. 5. *The intersection of any family of graded ideal is a graded ideal.*

For $(L_i)_{i\in I}$ be a family of graded ideals of A, then $\bigcap_{i\in I} L_i$ is an ideal in A. We need to check that $\bigcap_{i\in I} L_i$ is a graded submodule of A. If $x \in \bigcap L_i \subseteq A$, where $A = \Sigma_{\gamma\in\Gamma} A_\gamma$ (direct) then $x = \Sigma_{\gamma\in\Gamma} x_\gamma$ and $x_\gamma \in L_i$ for each i, since each L_i is graded; i. e. $x_\gamma \in \bigcap L_i$. Thus $\bigcap L_i$ is a graded submodule.

It is interesting to note here, that if S is any set, then the intersection of all graded ideals containing S is again a graded ideal containing S, and we call this the ideal generated by S. In this case the ideal J generated by S has at least two system of generators one non homogeneous as ideal, namely S and another homogeneous set of generators as graded ideal which we shall denote by $h(J)$; the existence of this is both necessary and sufficient for J to be graded [[1], theorem 4, p. 150].

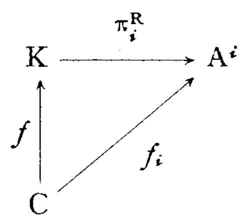
2. Existence of direct product.

Let $(A^i)_{i\in I}$ be a family of graded algebras in $\mathcal{G}_\Lambda^\Gamma$. Then their cartesian product $\prod_{i\in I} A^i$ is an algebra.

Now if the underlying graded module structure of each A^i is given by

$$A^i = \Sigma_{\gamma\in\Gamma} A^i_\gamma \quad \text{(direct)},$$

then the product $\prod_{i \in I} A_\gamma^i$ is obviously a submodule of $\prod_{i \in I} A^i$. Again the sum $K = \Sigma_{\gamma \in \Gamma} (\Pi A_\gamma^i)$ is a sumbodule and as easily seen, this sum is direct having Γ as its set of degrees. Now this is a subalgebra under the induced multiplication from ΠA^i. This subalgebra is thus a graded algebra, where the homogeneous elements of degree Γ are those of $\prod_{i \in I} A_\gamma^i$. We shall check that K is the direct product from the category theoretic view point [5] of the graded algebras A^i. To see this we denote by π_i^R the restriction of the natural projection $\pi_i : \Pi A^i \to A^i$ (which is an algebra homomorphism) to K then π_i^R is a graded algebra homomorphism. Again if $f_i : C \to A^i$ $(C \in \mathcal{G}_\Lambda^\Gamma)$ be a family of homomorphisms of graded algebras, then the map $f : C \to K$ defined by $f(c) = \Sigma_{\gamma \in \Gamma} [f_i(c_\gamma)]_{i \in I}$ where $c = \Sigma_{\gamma \in \Gamma} c_\gamma \in C$, is the unique graded algebra homomorphism making the diagram



commutative. This is an example where the direct product from the category theoretic view point do not coincide with the direct product in the usual sense.

3. Free product in $\mathcal{G}_\Lambda^\Gamma$.

Let an arbitrary family of Γ-graded Λ-algebras $(A^i)_{i \in I}$ be given. By a word we shall mean an ordered system of elements

(1) $$w = (a_2\, a_2 \cdots a_n)$$

where the length $n \geq 1$; every a_i is an element other than zero element of A^i; any two adjacent elements a_i, a_{i+1} belong to different A^i. If one of the component is zero, we shall denote

w by w_0 and call it a zero word. We identify the zero element of every algebra with w_0.

Now we consider the set of all words Ω. We make it a groupoid by introducing a binary operation of multiplication as follows

If $w' = (a'_1 \cdots a'_m)$ is another word then we define its product with w as

$$w w' = (a_1 \cdots a_n)(a'_1 \cdots a'_m)$$
$$= (a_1 \cdots a_n a'_1 \cdots a'_m)$$

If $a_n a'_1$ belongs to same A^i, then $a_n a'_1 = \bar{a}$, and

$$w w' = a_1 \cdots a_{n-1} \bar{a} a'_2 \cdots a'_m$$

i. e. to obtain the product we write these words in juxtraposition and contract once if necessary. Obviously $w w_0 = w_0$. It is easy to check the associative law for multiplication [6], so that Ω is infact a semi group.

Now in the set Ω, we pick up the homogeneous words $w_\gamma = a_{1\gamma_1} \cdots a_{n\gamma_n}$, where $a_{i\gamma_i} \in A^i_{\gamma_i}$; and each A^i has the direct sum decomposition $A^i = \sum_{\gamma_i \in \Gamma} A^i_{\gamma_i}$, such that $\gamma_1 + \gamma_2 + \cdots + \gamma_n = \gamma$.

We denote this subset by Ω_γ. We also assume that $w_0 \in \Omega_\gamma$ for each γ.

Immediately we have

LEMMA 3.1. *If* $w \in \Omega_\gamma$, *and* $w' \in \Omega'_\gamma$, *then* $w w' \in \Omega_{\gamma+\gamma'}$.

PROOF. We note that $l(w w') \leqq l(w) + l(w')$, where $l(w)$ = length of w. If $l(w w') = l(w) + l(w')$, this iso bvious. Again if $l(w w') \leqq l(w) + l(w')$ which occurs only when the last letter of w and the first letter of w' are from the same algebra A^i; then A^i being graded, we again have $w w' \in \Omega_{\gamma+\gamma'}$.

Next we consider the free Λ-module F on Ω, and introduce a multiplication in F, as follows, for $x \in F, y \in F$; we have

$$x = \sum_{w \in \Omega} \lambda_w^{(x)} w, \quad y = \sum_{w \in \Omega} \lambda_{w'}^{(y)} w'; \quad \lambda_w^{(x)}, \quad \lambda_{w'}^{(y)} \in \Lambda$$

and is equal to zero except for a finite number. Then we define $x y = \Sigma \lambda_w^{(x)} \lambda_{w'}^{(y)} (w w')$, where $w w'$ is defined in Ω as above.

This multiplication is associative, since the multiplication in Ω is associative, and also distributive; further this makes F an algebra, which we call the free product of graded algebras A^i, and we denote this by $F = {}^*\Pi A^i$.

Next we identify the elements $a_i \in A^i$ say with the word, (a_i) of length 1. Then by definition of the product in Ω, we have

$$w = (a_1 a_2 \cdots a_n) = (a_1)(a_2) \cdots (a_n) = a_1 a_2 \cdots a_n$$

LEMMA 3.2. *Every word* $w \in \Omega$, *is a sum of homogeneous words.*

PROOF. Let $w = (a_1 \cdots a_n)$, $a_i \in A^i$.

Then $a_1 = \sum_{\gamma \in \Gamma} a_{1\gamma}$ for $A^1 = \Sigma_{\gamma \in \Gamma} A^1_\gamma$; so in view of the above observation

$a_1 = (a_1) = \sum_{\gamma \in \Gamma} (a_{1\gamma}) \in F$ being the sum of homogeneous words of length 1, of the form $(a_{1\gamma})$ for $\gamma \in \Gamma$ with coefficient 1 from Λ. Now from the definition of multiplication

$$\begin{aligned} w = (a_1 \cdots a_n) &= (a_1)(a_2) \cdots (a_n) \\ &= \sum_{\gamma \in \Gamma} (a_{1\gamma}) \sum_{\gamma' \in \Gamma} (a_{2\gamma'}) \cdots \sum_{\bar\gamma \in \Gamma} (a_{n\bar\gamma}) \\ &= \sum_{\gamma, \gamma', \cdots, \bar\gamma} (a_{1\gamma})(a_{2\gamma'}) \cdots (a_{n\bar\gamma}) = \sum_{\gamma + \gamma' + \cdots + \bar\gamma} a_{1\gamma} a_{2\gamma'} \cdots a_{n\bar\gamma} \end{aligned}$$

Now the word $a_{1\gamma} a_{2\gamma'} \cdots a_{n\bar\gamma} \in \Omega_{\gamma + \gamma' + \cdots + \bar\gamma}$ in view of lemma 3.1. Hence we have lemma 3.2.

Next we denote by F_γ the submodule of F spanned by Ω_γ. Then by lemma 3.2. $F = \sum_{\gamma \in \Gamma} F_\gamma$ is the decomposition of F as the direct sum of submodules F_γ.

Any element of F has a unique representation

$$x = \sum_{w \in \Omega} \lambda_w w = \sum_{\gamma \in \Gamma} \Big(\sum_{w_\gamma \in \Omega_\gamma} \lambda w_\gamma \Big), \lambda \in \Lambda$$

and is zero except for a finite number. Using lemma 3.1., one can show that $F_\gamma{}_{\gamma'} \subseteq F_{\gamma + \gamma'}$. Thus the free product F is infact a graded algebra. The natural inclusion $\mu_i : A^i \to F$ is a homomorphism of graded algebra for each i. Next if $f_i : A^i \to X$ be a family of graded algebra homomorphism of A^i into $X \in \mathcal{G}_\Lambda^\Gamma$. Then there exist a unique homomorphism $f: F \to X$, such that the diagram

$$\begin{array}{ccc} A^i & \xrightarrow{\mu_i} & F = {}^*\prod A_i \\ & \searrow^{f_i} & \downarrow f \\ & & X \end{array} \tag{2}$$

commutes.

This f is defined by $f(w) = f_1(a_1) \cdots f_n(a_n)$, for $w = (a_1 \cdots a_n)$ from $\Omega \to X$ which extends immediately into the module homomorphism $F \to X$, and therefore to an algebra homomorphism; since $f(w\,w') = f(w)\,f(w')$. If $w_\gamma \in F_\gamma \subset F$, then we have

$$f(w_\gamma) = f\Bigl(\sum_{w'_\gamma \in \Omega_\gamma} \lambda_{w'} w'_\gamma\Bigr) = \sum_{w'_\gamma \in \Omega_\gamma} \lambda_{w'} f(w'_\gamma),$$

where w'_γ is of the form $a_{1_{\gamma_1}} a_{2_{\gamma_2}} \cdots a_{n_{\gamma_n}}$; $\gamma = \gamma_1 + \gamma_2 + \cdots + \gamma_n$ and $a_{i_{\gamma_i}}$ is homogeneous element of degree γ_i in A^i.

Now $f(w'_\gamma) = \prod_i f_i(a_{i_{\gamma_i}})$, f_i being graded àlgebra homomorphism preserves the degree and since $X \in \mathcal{G}_\Lambda^\Gamma$, $f(w_\gamma)$ has the same degree as w_γ; there exists at most one such f.

4. Free graded Algebra.

MacLane [7] defines a free graded Algebra as follows; let S be a set. Now we attach a degree to each element of S, arbitrarily from the commutative additive group Γ then we call S, a *graded set.*

Now if S is a graded set, we say a Γ-graded Algebra F over Λ, together with a grading preserving map $\Psi : S \to F$, a free graded algebra over S, if the following condition is satisfied.

Given any grading preserving map Φ of S into an algebra $A \in \mathcal{G}_\Lambda^\Gamma$, then there exists a unique graded algebra homomorphism $f: F \to A$, such that the diagram

$$\begin{array}{ccc} S & \xrightarrow{\Psi} & F \\ & \searrow^{\Phi} & \downarrow f \\ & & A \end{array} \qquad (3)$$

commutes.

Briefly we underline a construction of free graded algebras on S which is more simple then done by MacLane [7] for externally graded algebras.

We consider the free Λ-algebra (F, Ψ) over S which we know already exists. Then by definition of the free algebra the set $\{y^n = \Psi(x_{\gamma_1}) \cdots \Psi(x_{\gamma_n})\}$ where x_{γ_i} are elements of S with attached degree γ_i from Γ and $\Psi(x_{\gamma_1}) \cdots \Psi(x_{\gamma_n})$ is a finite composite of length n form a basis of F as a module over Λ. Now from this we pick up the sub set of all elements $\{y^n_\gamma = \Psi(x_{\gamma_1}) \cdots \Psi(x_{\gamma_n})\}$ where $\gamma = \gamma_1 + \gamma_2 + \cdots + \gamma_n \in \Gamma$. We denote by F_γ the module spanned by this subset. Then F is a direct sum $F = \sum_{\gamma \in \Gamma} F_\gamma$ such that Ψ is a grading preserving map. Evidently $F_\gamma \cdot F_{\gamma'} \subseteq F_{\gamma+\gamma'}$, because the length of the composite = sum of the length of the components.

This (F, Ψ) is a free graded algebra. For if Φ is any grading preserving function of S into a graded algebra A, then the unique algebra homomorphism $F \to A$ which exists in view of F being free, and makes the diagram (3) commutative is in fact a graded algebra homomorphism.

Corollary (1). *Given any Γ-graded Λ-algebra A, $\exists$ a free graded algebra F, with an epimorphism $F \to A \to 0$ of graded algebras.*

Proof. We denote by $h(A)$, the homogeneous elements of the algebra A, then the free algebra on the graded set $h(A)$ will have the required property.

Corollary (2). *If (F, ψ) is the free graded algebra on the graded set $\{x^{i_k}_{\gamma_k}\}^{\gamma_k \in \Gamma}_{i_k \in I}$ and (F_{i_l}, ψ_{i_l}) is the free graded algebra on the graded singleton $\{x^{i_l}_{\gamma_l}\}$, then for the free product $^*\Pi F_{i_l}$; $F \simeq \Pi^* F_{i_l}$.*

5. Projective graded Algebras.

DEFINITION 5.1. A graded algebra is said to be *projective* if every diagram of graded algebras and graded algebra homomorphisms

$$\begin{array}{ccccc} & & A & & \\ & & \downarrow & & \\ C & \longrightarrow & B & \longrightarrow & 0 \end{array}$$

in which the row is exact can be completed to a commutative diagram, of graded algebras and graded algebra homomorphism,

$$\begin{array}{ccccc} & & A & & \\ & \theta\swarrow & \downarrow\varphi & & \\ C & \underset{\psi}{\longrightarrow} & B & \longrightarrow & 0. \end{array}$$

As usual one obtains

THEOREM 5.2. (i) *The free product* $\Pi^* A^\alpha$ *of graded algebras* A^α *is projective if and only if* A^α *is projective for all* α.

(ii) *A free graded algebra* (F, ψ) *is projective.*

PROOF. The proofs are straight forward.

DEFINITION 5.3. We call a graded subalgebra B of a graded algebra A, a *graded rectract,* if $B =$ image of g, where g is an idempotent graded algebra endomorphism of A; As in the module theory we again have

THEOREM 5.3. (i) *A graded retract of a projective graded algebra is projective.*

(ii) *Every projective graded algebra is isomorphic to a graded retract of a free graded algebra.*

Definition 5.4. Let A be Γ-graded; A *coordinate system* for A is a set I, a graded set $\{a^{\alpha}_{\gamma}\}^{\gamma \in \Gamma}_{\alpha \in J}$ of elements of A indexed by a set J, and a family of elements $\{\varphi_i\}_{i \in I}$ of $\mathrm{Hom}_{\Lambda}(A, \Lambda)$, the module of all linear mappings of A into Λ (considered as module), such that

i) for $a \in A$, $\{i \in I$, such that $\varphi_i(a) \neq 0$ is finite$\}$

ii) for $a \in A$, $a = \Sigma^{\gamma \in \Gamma}_{i \in I} \varphi_i(a)\ y^{k(i)}_{\gamma}$ where $k : I \to K$

is a mapping and K consists of all finite sequences of elements of J, such that

$$y^{k(i)}_{\gamma} = a^{\alpha}_{\gamma'},\ a^{\beta}_{\gamma''} \cdots a^{\varepsilon}_{\gamma'''} \cdots \quad \text{where} \quad \gamma' + \gamma'' + \cdots + \gamma''' \cdots = \gamma$$

and $k(i) = (\alpha, \beta, \cdots \varepsilon)$. [We note that K is in fact a semigroup]. As in module theory, one obtains

Theorem 5.4. *A graded algebra* A *is projective if and only if there exists a coordinate system for* A.

Proof follows using Theorem 5.3. (*i*) and Theorem 5.2. (*ii*) for the 1st part, second part is obvious.

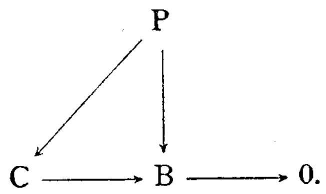
6. Varieties of graded algebras.

The theory of graded algebras was important because every algebra is in fact a graded algebra with trivial gradation. In this section we study varieties of graded algebras and generalize Fröhlich's work on groupes over d. g. near rings to graded algebras [4]. In the sequel we shall denote by $\mathrm{hom}^{\Gamma}_{\Lambda}(A, B)$, the set of all graded algebra homomorphism from A to B, also $\mathcal{C}_1$ will denote a subcategory of $\mathcal{G}^{\Gamma}_{\Lambda}$.

Definition 6.1. A graded algebra P will be called *projective for* $\mathcal{C}_1$, if every diagram

$$\begin{array}{ccccc} & & P & & \\ & & \downarrow & & \\ C & \longrightarrow & B & \longrightarrow & 0 \end{array}$$

of graded algebra homomorphisms with the row exact and with $C, B \in \mathcal{C}_1$ can be completed to a commutative diagram in $\mathcal{G}_\Lambda^\Gamma$,



If in addition $P \in \mathcal{C}_1$, we shall say that P is $\mathcal{C}_1$*-projective.*

DEFINITION 6. 2. A *free basis for* $\mathcal{C}_1$ (free $\mathcal{C}_1$-basis) in a graded algebra F is a graded set S of F, such that for every graded algebra $A \in \mathcal{C}_1$ and for every grading preserving map $\varphi' : S \to A$, there exists a unique graded algebra homomorphism $\varphi : F \to A$, extending φ'. If F has a free $\mathcal{C}_1$-basis, then F is free for $\mathcal{C}_1$. Again if further $F \in \mathcal{C}_1$, we call F, $\mathcal{C}_1$*-free.*

We shall consider subfunctor and quotient functor of a covariant functor from $\mathcal{G}_\Lambda^\Gamma$ *, in the sense of EILENBERG & MACLANE [3].

If G is a covariant functor of $\mathcal{G}_\Lambda^\Gamma$ and if the algebra G(A) is always a graded subalgebra of F(A) and if $G(\alpha)$ is given by the restriction of $F(\alpha)$ to G(A), then we call G a *subfunctor* of F. If moreover G(A) is always a graded ideal of F(A), then G is called a *normal subfunctor* of F. In the later case the quotient graded algebra F(A)/G(A) defines in the natural manner a quotient functor F/G. To define a subfunctor G of F, it will suffice to specify the graded subalgebras G(A) of F(A), provided it can be verified that for a graded algebra homomorphism $\alpha : A \to B$, $F(\alpha)$ maps $G(A) \to G(B)$.

Let G_1, G_2 be normal subfunctors of F, and we put

(*i*) $G_1 + G_2(A) = G_1(A) + G_2(A)$ i. e. the ideal generated by the $h(G_1(A)) \cup h(G_2(A))$ where $h(G)$ has meaning as in proposition 1. 5.

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* We prefer not to specify the co domain category. The reades may take it as another category $\mathcal{G}_{\Lambda'}^{\Gamma'}$, though our interest lies in functors taking values again in $\mathcal{G}_\Lambda^\Gamma$. From the definition it is clear that the property of preserving monomorphisms (epimorphisms) is inherited by subfunctors (quotient functors).

(*ii*) $G_1 \cap G_2(A) = G_1(A) \cap G_2(A)$

(*iii*) $G_1 G_2(A) = G_1(A) G_2(A)$

(*iv*) $[G_1, G_2] A = G_1(A) G_2(A) + G_2(A) G_1(A)$, sum being defined as in (*i*).

PROPOSITION 6. 3. $G_1 + G_2$, $G_1 \cap G_2$, $G_1 G_2$, $[G_1, G_2]$ *are all normal subfunctors of* F.

DEFINITION 6. 4. A full subcategory $\mathcal{C}$ of $\mathcal{G}_A^\Gamma$ is called a *variety* if

(*i*) $f: A \rightarrow B$ an epimorphism of graded algebras $A \in \mathcal{C}$ $\Rightarrow B \in \mathcal{C}$ (Q-closed)

(*ii*) $f: B \rightarrow A$ a monomorphism of graded algebras $A \in \mathcal{C}$ $\Rightarrow B \in \mathcal{C}$ (S-closed).

(*iii*) For an indexed set of graded algebras $\{A^\mu\}$ of $\mathcal{C}$, the direct product $\Pi A^\mu \in \mathcal{C}$ (Π-closed). [§ 2].

As a trivial example we consider all algebras as a graded algebras by attaching degree 0 to each of its element ($0 \in \Gamma$ is the identity element of Γ) then the algebras form a variety $\mathcal{C}$ of $\mathcal{G}_A^\Gamma$. Again in the category of algebras, the modules form a variety with trivial multiplication.

DEFINITION 6. 5. A *covariant functor* V of $\mathcal{G}_A^\Gamma$ is called a *variety functor*

(*i*) if V is normal subfunctor of the identity functor I,

(*ii*) V preserves epimorphisms.

PROPOSITION 6. 6. *If* V_1, V_2 *are variety functors, then so are* $V_1 + V_2$, (V_1, V_2) *and* $V_1 V_2$.

Proof is immediate using proposition 6. 3.

Analogus to the groups over near rings, we have.

PROPOSITION 6. 7. *A normal subfunctor* V *of* I *is a variety functor if and only if the quotient functor* I/V *is right exact*

The biunique correspondence between varieties and variety functors go over to $\mathcal{G}_A^\Gamma$ in the following form.

THEOREM 6.8. (i) *If* V *is variety functor, then the graded algebras* A, *with* $V(A)=0$ *form a variety* $\mathcal{C}_V$.

(ii) *If* $\mathcal{C}$ *is a variety, then every graded algebra* A *has a minimal graded ideal* $V(A)$ *such that* $A/V(A)\in\mathcal{C}$. *These ideals determines a subfunctor* $V_{\mathcal{C}}$ *of* I *and this is a variety functor.*

(iii) $V=V_{\mathcal{C}}\Longleftrightarrow\mathcal{C}=\mathcal{C}_V$.

To check that $\mathcal{C}_V$ is S-closed and Q-closed is easy. Now let A be the direct product of the family $(A^i)_{i\in I}$ of graded algebras $\in\mathcal{C}_V$. Then $A=\Sigma_{\gamma\in\Gamma}(\Pi A^i_\gamma)_{i\in I}$ with the projections $\Pi^R_i\, A\to A_i$. [Note $A^i=\Sigma_{\gamma\in\Gamma}A^i_\gamma$]. By hypothesis $V(\Pi^R_i)$ is null for all i. But $V(\Pi^R_i)$ is the restriction of Π^R_i to $V(A)$. Hence $V(A)\subseteq \bigcap\operatorname{Ker}\pi^R_i=0\Rightarrow V(A)=0$ i. e. $A\in\mathcal{C}_V$.

So $\mathcal{C}_V$ is a variety.

(*ii*) Suppose $\mathcal{C}$ is a variety, for every graded algebra A, let $V(A)$ be the intersection of all graded ideals J_λ such that $A/J_\lambda\in\mathcal{C}$. Then $V(A)$ is a graded ideal of A. That $V(A)$ is minimal and epifunctorial is easy to verify.

(*iii*) is easy to check.

Using the fact that every algebra $A\in\mathcal{G}^\Gamma_\Lambda$ has a representation by an exact sequence $F\to A\to 0$ where F is $\mathcal{G}^\Gamma_\Lambda$ free, we get for any variety $\mathcal{C}$, with associated variety functor V, the

PROPOSITION 6.9. (i) *The* $\mathcal{C}$ *free algebras* A, *are precisely those algebras which have a representation by an exact sequence*

$$0\to V(F)\to F\to A\to 0 \text{ where } F \text{ is } \mathcal{G}^\Gamma_\Lambda\text{-free}.$$

(ii) *The algebras* A *in* $\mathcal{C}$, *are precisely those, which have a representation by an exact sequense*

$$\overline{F}\to A\to 0 \text{ where } \overline{F} \text{ is } \mathcal{C}\text{-free}.$$

(iii) *The* $\mathcal{C}$*-projective algebras are isomorphic to the graded retracts of* $\mathcal{C}$*-free algebras.*

We are content to leave the proof for the reader.

By Proposition 6.9 the variety $\mathcal{C}$ is completely determined by the algebras $V(F)$ where F is $\mathcal{G}^\Gamma_\Lambda$ free. We proceed now to check that the classical characterisation of varieties interms of fully invariant subgroups can be carried over.

DEFINITION 6.10. A graded ideal $J \subseteq A$ is called a *fully invariant graded ideal,* if it is mapped into itself under all graded algebra endomorphisms of A.

PROPOSITION 6.11. (i) *If* V *is a variety functor and* $A \in \mathcal{G}^{\Gamma}_{\Lambda}$, *then* $V(A)$ *is fully invariant graded ideal of* A.

(ii) *A fully invariant graded ideal* J *of a* $\mathcal{G}^{\Gamma}_{\Lambda}$-projective *graded algebra* A, determines a variety $\mathcal{C}$, by the law $B \in \mathcal{C}$, *if* $J \subseteq \operatorname{Ker} \alpha$ *for all* $\alpha \in \hom^{\Gamma}_{\Lambda}(A, B)$. *Also* $J = V_{\mathcal{C}}(A)$.

The proofs are straight forward.

EXAMPLES OF VARIETIES: Let $\mathcal{G}^{Z}_{\Lambda}$ be the category of Z-graded Λ-algebras, where Z is the ring of integers.

Then any $A \in \mathcal{G}^{Z}_{\Lambda}$ has the form

$$A = \sum_{-\infty}^{\infty} A_n .$$

We take

(*i*) $\mathcal{C}_1 = \{A \in \mathcal{G}^{Z}_{\Lambda} \mid A_n = 0 \text{ for all } n\}$

(*ii*) $\mathcal{C}_2 = \{A \in \mathcal{G}^{Z}_{\Lambda} \mid A_n = 0 \text{ for all } n < 0\}$

(*iii*) $\mathcal{C}_3 = \{A \in \mathcal{G}^{Z}_{\Lambda} \mid A_n = 0 \text{ for all } n \leq 0\}$

(*iv*) $\mathcal{C}_4 = \{A \mid \in \mathcal{G}^{Z}_{\Lambda} \mid A_n = 0 \text{ for } n \neq 0\}$.

Then $\mathcal{C}_1$, $\mathcal{C}_2$, $\mathcal{C}_3$, $\mathcal{C}_4$ are varieties.

(*v*) In $\mathcal{G}^{\Gamma}_{\Lambda}$, let us consider any subset Σ of Γ, then the graded algebras $\mathcal{C} = \{A \mid A_{\sigma} = 0 \text{ for } \sigma \in \Sigma\}$ form a variety.

7. Covarieties.

Dual concept of a variety is that of a covariety.

DEFINITION 7. 1. A *full* subcategory $\mathcal{F}$ of $\mathcal{G}_\Lambda^\Gamma$ is called a *covariety* if it satisfies closure property (*i*), (*ii*), in definition 6. 4 for $\mathcal{F}$ with the additional condition

(*iii*) The free product $\Pi^* A^\mu$ of a family of indexed set algebras A^μ from $\mathcal{F}$, lies in $\mathcal{F}$.

DEFINITION 7. 2. A covariant functor F of $\mathcal{G}_\Lambda^\Gamma$ is called a *covariety functor* if

(*i*) F is a subfunctor of I.
(*ii*) Whenever $\alpha: B \to A$ is a monomorphism in $\mathcal{G}_\Lambda^\Gamma$, then image of $F(\alpha) = \text{Image } \alpha \cap F(A)$.

The covariety functor satisfies the dual law of 6. 7.

PROPOSITION 7. 3. *A covariety functor is left exact.*

Mirror image of proposition 6. 8. will be,

PROPOSITION 7. 4. (i) *If* F *is a covariety functor, then the graded algebras* A, *with* $F(A) = A$ *form a covariety* $\mathcal{F}^F$.
(ii) *If* $\mathcal{F}$ *is a covariety, then every algebra* A *has a unique maximal graded subalgebra* $F^{\mathcal{F}}(A)$ *lying in* $\mathcal{F}$. *These subalgebras* $F^{\mathcal{F}}(A)$ *define a subfunctor* $F^{\mathcal{F}}$ *of* I, *and this is a covariety functor.*
(iii) $F = F^{\mathcal{F}} \Longleftrightarrow \mathcal{F} = \mathcal{F}^F$.

EXAMPLES OF COVARIETIES:

In $\mathcal{G}_\Lambda^Z$,

(*i*) $\mathcal{F}_1 = \{A \mid A_n = 0 \text{ for all } n\}$
(*ii*) $\mathcal{F}_2 = \{A \mid A_n = 0 \text{ for } n \neq 0\}$.

Then $\mathcal{F}_1$ and $\mathcal{F}_2$ are covarieties. We note, the intersection of varieties and covarieties could be nonempty as in example (*i*), (*ii*).

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Added in PROOF: Since this paper was written long time ago (1966), various generalizations of these concepts in terms of categories was done by the authors and others. However it still contains many original and interesting results.