PORTUGALIAE MATHEMATICA

VOLUME 29

1 9 7 0

Edição de

«GAZETA DE MATEMÁTICA, LDA»

PORTUGALIAE MATHEMATICA
Rua Diário de Notícias, 134, 1.º-Esq.
LISBOA - 2 (PORTUGAL)

PORTUGALIAE MATHEMATICA
Vol. 29 — Fasc. 1 — 1970

ON A LOCAL PROPERTY OF $|(N, p_n)(C, \alpha)|$ SUMMABILITY OF FOURIER SERIES (1)

BY FU CHENG HSIANG
National Taiwan University Formosa — China

1. Let Σa_n be a given infinite series with the sequence of partial sums $\{S_n\}$ and let $\{p_n\}$ be a sequence of non-negative and monotonic non-increasing sequence of constants such that $p_0 > 0$ and $P_n = p_0 + p_1 + p_2 + \cdots + p_n \to \infty$. If the sequence of transformation given by

$$t_n = \frac{1}{P_n} \sum_{\nu=0}^{n} p_{n-\nu} S_\nu$$

is of bounded variation, i. e., $\Sigma |t_n - t_{n-1}| < \infty$, then we say that the series or the sequence $\{S_n\}$ is absolutely summable (N, p_n) or summable $|N, p_n|$.

Let σ_n^α denote the n-th Cesàro means of order α of Σa_n. If the sequence of transformation given by

$$\tau_n^\alpha = \frac{1}{P_n} \sum_{\nu=0}^{n} p_{n-\nu} \sigma_\nu^\alpha$$

is of bounded variation, then Σa_n is said to be summable $|(N, p_n)(C, \alpha)|$. It is known that [1] if $\Sigma |S_n|/P_n < \infty$, then Σa_n is summable $|N, p_n|$. Consequently, we obtain the following corresponding theorem for $|(N, p_n)(C, \alpha)|$ summability of Σa_n.

LEMMA 1. *If* $\Sigma |\sigma_n^\alpha|/P_n < \infty$, *then* Σa_n *is summable* $|(N, p_n)(C, \alpha)|$.

(1) Received January, 1968.

We denote by

$$D_\nu(t)=\frac{1}{2}\sum_{\mu=1}^{\nu}\cos\mu t=\frac{\sin(\nu+1/2)t}{2\sin(t/2)}$$

the ν-th DIRICHLET kernel, and write

$$A_\nu^\alpha=\binom{\nu+\alpha}{\nu}=\frac{(\alpha+1)(\alpha+2)\cdots(\alpha+\nu)}{\nu!}\ (\alpha>-1),$$

$$K_n^\alpha(t)=\frac{1}{A_\nu^\alpha}\sum_{\nu=0}^{n}A_{n-\nu}^{\alpha-1}D_\nu(t).$$

We have the following known estimation for $K_n^\alpha(t)$.

LEMMA 2. [2, p. 90, (3. 10); p. 94, (5. 5)]. *For* $0<t<\pi$,

$$|K_n^\alpha(t)|\leqq A_\alpha n^{-\alpha}t^{-(\alpha+1)}\ (0\leqq\alpha\leqq1),$$

where A_α *is an absolute constant depending only on* α.

2. Let $\sigma_n^\alpha(x)$ denote the n-th Cesàro means of order α of the FOURIER series of $f(t)$ at $t=x$, then

$$\sigma_n^\alpha(x)=\frac{2}{\pi}\int_0^\pi\varphi_x(t)K_n^\alpha(t)\,dt,$$

where

$$\varphi_x(t)=\frac{1}{2}\Big\{f(x+t)-f(x-t)\Big\}.$$

We prove a theorem for the local property of $|(N,p_n)(C,\alpha)|$ summability of the FOURIER series of $f(t)$ at $t=x$.

THEOREM. *Let* $\{p_n\}$ *be a non-negative and monotonic non-increasing sequence of constants such that* $p_0>0$ *and* $P_n\to\infty$. *If* $\Sigma 1/n^\alpha P_n<\infty\,(0\leqq\alpha\leqq1)$, *then* $|(N,P_n)(C,\alpha)|$ *summability of the* FOURIER *series of* $f(t)$ *at* $t=x$ *depends only on the local property of* $f(t)$ *near the point* $t=x$.

3. We now prove the theorem. Write, for an arbitrary $0 < \delta < \pi$,

$$\sigma_n^\alpha(x) = \frac{2}{\pi}\left(\int_0^\delta + \int_\delta^\pi\right) \varphi_x(t)\, K_n^\alpha(t)\, dt = \frac{2}{\pi}(I_{1n} + I_{2n}),$$

say. By Lemma 1, we have to establish

$$\sum_{n=0}^{\infty} \frac{|I_{2n}|}{P_n} \leq \sum_{n=1}^{\infty} \frac{1}{P_n}\left|\int_\delta^\pi \varphi_x(t)\, K_n^\alpha(t)\, dt\right| + A < \infty .$$

By Lemma 2, we have

$$\left|\int_\delta^\pi \varphi_x(t)\, K_n^\alpha(t)\, dt\right| \leq A_\alpha n^{-\alpha} \int_\delta^\pi |\varphi_x(t)|\, t^{-(\alpha+1)}\, dt$$

$$\leq B_\alpha n^{-\alpha} \int_\delta^\pi |\varphi_x(t)|\, dt \leq C_\alpha n^{-\alpha} .$$

Thus,

$$\sum_{n=1}^{\infty} \frac{|I_{2n}|}{P_n} \leq C_\alpha \sum_{n=1}^{\infty} \frac{1}{n^\alpha P_n} < \infty .$$

Since the integrals I_{1n} depend only the local property of near the point $t = x$, the theorem follows.

REFERENCES

[1] S. N. BHATT, *An aspect of local property of absolute Nörlund summability of a Fourier series,* Proc. Nat. Inst. Sci. of India, **28** (1962), 787-794.

[2] A. ZYGMUND, *Trigonometric Series,* Vol. I, Cambridge, 1959.