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REMARK ON A FORMULA OF P. H. JACKSON (1)

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In a paper [5] that appeared in this Journal in 1941 JACKSON proved three very interesting  $q$ -identities. The second of these is the following formula:

$$(1) \quad \sum_{r=0}^{2n} (-1)^r \binom{2n}{r} (a)_r (b)_r (a)_{2n-r} (b)_{2n-r} q^{r(2n-r+1)/2} \\
 = (a)_n (b)_n (q^{n+1})_n (a b q^n)_n,$$

where

$$(a)_n = (1-a)(1-qa)\cdots(1-q^{n-1}a),$$

$$\binom{2n}{r} = \frac{(q)_{2n}}{(q)_r (q)_{2n-r}};$$

the parameters  $a, b$  are unrestricted. See also BAILEY [1].

JACKSON remarks: «a special case of (1), with an *unusual solitary factor*, is

$$(2) \quad 1 + \sum_{r=1}^{2n} (-1)^r q^{r(2n-r+1)/2} = (1-q^{2n})(1-q^{2n-1})\cdots(1-q^{n+1}).$$

Dr. W. N. BAILEY kindly drew my attention to this curious identity for an *apparently simple product*.»

It has probably been noticed that (2) is not correct. Indeed it is evident that

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$$\begin{aligned} S &= \sum_{r=1}^{2n} (-1)^r q^{r(2n-r+1)/2} \\ &= \sum_{r=1}^{2n} (-1)^{2n-r+1} q^{(2n-r+1)r/2} = -S, \end{aligned}$$

so that  $S = 0$ .

If we take  $a = b = 0$  in (2) we get

$$(3) \quad \sum_{r=0}^{2n} (-1)^r \binom{2n}{r} q^{r(2n-r+1)/2} = (q^{n+1})_n,$$

and it may be that this is what BAILEY and JACKSON had in mind.

The identity (3) is reminiscent of the well known identity

$$(4) \quad \sum_{r=0}^{2n} (-1)^r \binom{2n}{r} = (1-q)(1-q^3) \cdots (1-q^{2n-1}),$$

which GAUSS employed in evaluating the Gaussian sum [2].

We remark that (4) can be proved rapidly as follows. Recalling the identity

$$e(z, q) = \prod_0^{\infty} (1 - q^n z)^{-1} = \sum_0^{\infty} z^n / (q)_n,$$

it follows that

$$\begin{aligned} e(z, q) e(-z, q) &= e(z^2, q^2) \\ &= \sum_{n=0}^{\infty} \frac{z^{2n}}{(q)_{2n}} \sum_{r=0}^{2n} (-1)^r \binom{2n}{r}. \end{aligned}$$

Comparison of coefficients yields (4).

Proof of (3) is somewhat less simple. We have

$$(5) \quad \sum_{r=0}^{2n} (-1)^r \binom{2n}{r} q^{r(2n-r+1)/2} = q^{n(n+1)/2} S_{2n},$$

where

$$S_{2n} = \sum_{r=-n}^n (-1)^{n-r} \binom{2n}{n-r} q^{-r(r+1)/2}.$$

Then

$$\begin{aligned}
 S_{2n} &= \sum_r (-1)^{n-r} \left\{ \binom{2n-1}{n-r} + \binom{2n-1}{n-r-1} q^{n+r} \right\} q^{-r(r+1)/2} \\
 &= \sum_r (-1)^{n-r} \binom{2n-1}{n-r} q^{-r(r+1)/2} \\
 &\quad + q^n \sum_r (-1)^{n-r} \binom{2n-1}{n-r-1} q^{-r(r-1)/2} \\
 &= (1 + q^n) S_{2n-1},
 \end{aligned}$$

where

$$S_{2n-1} = \sum_{r=-n}^{n-1} (-1)^{n-r} \binom{2n-1}{n+r} q^{-r(r-1)/2}.$$

On the other hand

$$\begin{aligned}
 (1 - q^{2n-1}) S_{2n-2} &= (1 - q^{2n-1}) \sum_r (-1)^{n-r-1} \binom{2n-2}{n+r-1} q^{-r(r+1)/2} \\
 &= \sum_r (-1)^{n-r-1} \binom{2n-1}{n+r} (1 - q^{n+r}) q^{-r(r+1)/2} \\
 &= \sum_r (-1)^{n-r-1} \binom{2n-1}{n+r} q^{-r(r+1)/2} \\
 &\quad + q^n \sum_r (-1)^{n-r} \binom{2n-1}{n+r} q^{-r(r-1)/2}.
 \end{aligned}$$

Since

$$\begin{aligned}
 S &= \sum_r (-1)^{n-r} \binom{2n-1}{n+r} q^{-r(r+1)/2} \\
 &= \sum_r (-1)^{n-r} \binom{2n-1}{n-r-1} q^{-r(r+1)/2} \\
 &= - \sum_r (-1)^{n-r} \binom{2n-1}{n-r} q^{-r(r-1)/2} \\
 &= - \sum_r (-1)^{n-r} \binom{2n-1}{n+r} q^{-r(r+1)/2} = -S,
 \end{aligned}$$

it follows that

$$(1 - q^{2n-1})S_{2n-2} = q^n S_{2n-1}.$$

Hence

$$S_{2n} = q^{-n}(1 + q^n)(1 - q^{2n-1})S_{2n-2},$$

which implies

$$(6) \quad S_{2n} = q^{-n(n+1)/2} (q^{n+1})_n.$$

Combining (5) and (6), we get (3).

If we replace  $q$  by  $q^{-1}$  we find that (3) becomes

$$(7) \quad \sum_{r=-n}^n (-1)^r \binom{2n}{n-r} q^{r(3r+1)/2} = (q^{n+1})_n.$$

We remark that (7) is equivalent to formula (6.16.5) of [4], a formula used in proving the ROGERS-RAMANUJAN identities. In this connection see also [2].

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