PORTUGALIAE MATHEMATICA

VOLUME 32

1973

Edição de

«GAZETA DE MATEMÁTICA, LDA»

PORTUGALIAE MATHEMATICA
Rua Diário de Notícias, 134, 1.º-Esq.
LISBOA - 2 (PORTUGAL)

PORTUGALIAE MATHEMATICA
Vol. 32 — Fasc. 3 — 1973

α-POINT COMPACTIFICATIONS (1)

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Introduction. From the work of CECH [1] and STONE [4] it is known that if Y is a compactification of X then cardinal $Y = |Y| \leq |\beta X|$ and $|Y \setminus X| \leq |\beta X \setminus X|$ where βX is the STONE-CECH compactification of X. In a sense, more points must be "added" to X to get βX than to obtain any other compactification of X and in case $X = N$, the countably infinite discrete space, $|\beta X| = 2^c$ where c is the power of the continuum. In the construction of βX by GILLMAN and JERISON [2] the points of $\beta X \setminus X$ are the free ultrafilters from $Z(X) =$ the lattice of zero sets of X. In this paper we show that for any cardinal number α there is a topological space X such that $|\beta X \setminus X| = \alpha$. Equivalently, there are α free ultrafilters from $Z(X)$. Call a space S a *residue* of X if there is a compactification Y of X such that $Y \setminus X \approx S$. We show that every compact separable HAUSDORFF space is a residue of N from which it follows that every such is a continuous image of $\beta N \setminus N$ and has cardinal at most 2^c.

Define Y a *compactification* of X if Y is a compact HAUSDORFF space and X is dense in Y. Necessarily X is completely regular and T_1. Let $C^*(X)$ be the collection of bounded real-valued continuous functions defined on X and say X is *C^*-embedded* in Y in case every $f \in C^*(X)$ has a continuous extension to Y. Define Y an *α-point* compactification of X in case $|Y \setminus X| = \alpha$. The unit interval, $[0,1]$, is an $\aleph_0$-point compactification of the irrationals in $[0,1]$. HAUSDORFF [3] showed that βN is a 2^c-point compactification of N, i. e., the power set of N, which is $Z(N)$, contains 2^c free ultrafilters.

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(1) Received January, 1970.

We state two properties of βX:

1) βX is the compactification of X in which X is C^*-embedded, i. e., if Y is a compactification of X in which X is C^*-embedded then Y and βX are homeomorphic by a map that leaves X pointwise fixed.

2) If f is a continuous functiom from X to Y, Y compact, then f has a continuous extension to βX. In particular, if Y is a compactification of X then the identity function $i: X \to X$ extends to $\hat{i}: \beta X \to Y$ onto and as a result $|Y \setminus X| \leq |\beta X \setminus X|$.

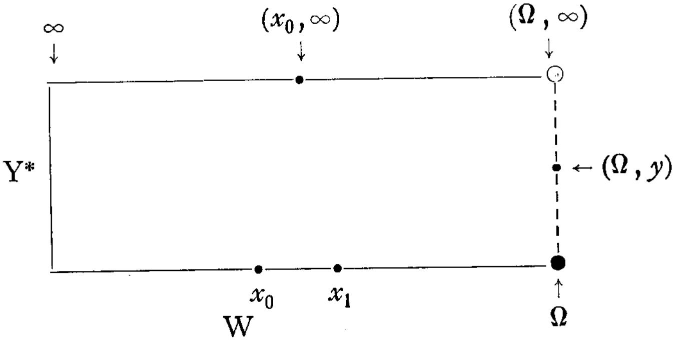
Let W be the space of ordinal numbers strictly less than the first uncountable one, Ω, with the order topology. Every $f \in C(W)$ is constant on a tail of W as is well known.

THEOREM 1. *For every cardinal number* α *there is a completely regular,* T_1 *space* X *such that* βX *is an* α*-point compactification of* X.

PROOF. Take α infinite. Let $W =$ all ordinal numbers less than the 1st uncountable one, which is Ω, with the order topology.

Let $W^* = W \cup \{\Omega\}$ and take Y to be the discrete space of cardinal α and Y^* its 1-pt. compactification. $Y^* = Y \cup \{\infty\}$.

Set $X = W \times Y^*$. I claim βX is obtained by adding $\{\Omega\} \times Y$ to X, or by the diagram, adding the right edge.



To do this take $f \in C^*(X)$ and extend f continuously to $W^* \times Y^*$. For $y \in Y$, $W \times \{y\} \approx W$ so on $W \times \{y\}$, f is constant on a tail: define $\hat{f}(\Omega, y) = c_y$, this constant. In particular for $y=\infty$ set $\hat{f}(\Omega,\infty)=c_\infty$. It is easy to see $\hat{f}$ is continuous at (Ω,y) for $y\neq\infty$. Take 0 open in the Reals, $\hat{f}(\Omega,y)=c_y\in 0$. Pick $\alpha\in W$ with $(\alpha,\Omega]\times\{y\}$ on the tail of $W\times\{y\}$ which f takes to c_y. Since $\{y\}$ is open in Y, $(\alpha,\Omega]\times\{y\}$ is an open set in $W^*\times Y^*$ containing the point (Ω,y) and $\hat{f}((\alpha,\Omega]\times\{y\})=\{c_y\}\subset 0$. We have extended f continuously to $W^*\times Y^*\setminus\{(\Omega,\infty)\}$.

To prove $\hat{f}$ is continuous at (Ω,∞) where $\hat{f}(\Omega,\infty)=c_\infty$ take 0 open in the Reals, $\hat{f}(\Omega,\infty)=c_\infty\in 0$. We want to find G open in $W^*\times Y^*$, $(\Omega,\infty)\in G$ and $\hat{f}(G)\subset 0$. Assume no G exists, that is, for every G open, $(\Omega,\infty)\in G$ there is $(g_1,g_2)\in G$ with $\hat{f}(g_1,g_2)\notin 0$.

First, pick $x_0\in W$ such that $\hat{f}((x_0,\Omega]\times\{\infty\})=c_\infty$. Now take an open set $(x_0,\Omega]\times A$ containing (Ω,∞). Now $Y^*\setminus A$ is finite so there must be $(x_1,y_1)\in(x_0,\Omega]\times A$ with $\hat{f}(x_1,y_1)\notin 0$. Next take $(x_1,\Omega]\times A_1$ open, containing (Ω,∞) with $y_1\notin A_1$. There must be $(x_2,y_2)\in(x_1,\Omega]\times A_1$ with $f(x_2,y_2)\notin 0$. Note: $x_1\neq x_2$, $y_1\neq y_2$, $x_1<x_2$.

Note: I prefer to keep off the right edge [i. e., $x_1=\Omega$ or $x_2=\Omega$] so if $x_2=\Omega$ just move it off the edge to a point (x'_2,y_2) which is on the tail of $W\times\{y_2\}$ that f takes to c_{y_2}. So we still have $f(x'_2,y_2)\notin 0$ with $(x'_2,y_2)\in(x_1,\Omega]\times A_1$.

Continuing inductively obtain a sequence $\langle(x_i,y_i)\rangle_{i=1}^{\infty}$, distinct and $i\neq j$ implies $x_i\neq x_j$ and $y_i\neq y_j$. Also $i<j$ implies $x_i<x_j$, $\{y_i\}_{i=1}^{\infty}$ is an infinite set. As a result $\langle x_i\rangle_{i=1}^{\infty}\Rightarrow\sup_i x_i\equiv x<\Omega$ and $\langle y_i\rangle_{i=1}^{\infty}\Rightarrow\infty$. Therefore $\langle(x_i,y_i)\rangle_{i=1}^{\infty}\Rightarrow(x,\infty)$ and since $f\in C(X)$ we have $\langle f(x_i,y_i)\rangle_{i=1}^{\infty}\Rightarrow f(x,\infty)=c_\infty\in 0$ $[x_0<x]$ which contradicts our assumption that $f(x_i,y_i)\notin 0$ for every i. Therefore $\hat{f}\in C^*(W^*\times Y^*)$ and $\hat{f}|_X=f$ and X is C^*-embedded in the compact, T_2-space $W^*\times Y^*$. By uniqueness of βX get $\beta X=W^*\times Y^*$.

For α a finite cardinal let $X=W\times Y$ where Y is the discrete space with cardinal α.

Consider $[0,1]\times\{0\}\approx[0,1]$. We now exhibit a compactification Y of N such that $Y\setminus N\approx[0,1]$. For $n\in N$ define $A_n=\left\{\left(\frac{k}{2^n},\frac{1}{n}\right):k\in N,\ 1\leq k\leq 2^n-1\right\}$ and notice that $\bigcup_{n=1}^{\infty}A_n\approx N$. Set $Y=\left(\bigcup_{n=1}^{\infty}A_n\right)\cup[0,1]$ which is a c-point compactification of N. In the study of residues of N we found a large class of spaces that are residues. This class contains the compact metric spaces.

THEOREM 2. *If* X *is a compact separable* T_2 *space then* X *is a residue of* N.

PROOF. Consider $X \times [0,1]$ and embed X in this space as $X \times \{0\}$. Let $D = \{p_1, p_2, \cdots\}$ be a countable dense subspace of X. Define for $n \in N$, $A_n = \left\{\left(p_1, \frac{1}{n}\right), \left(p_2, \frac{1}{n}\right), \cdots, \left(p_n, \frac{1}{n}\right)\right\}$.

CLAIM. $N \approx \bigcup_{n=1}^{\infty} A_n$. To see this take $\left(p_i, \frac{1}{n}\right) \in \bigcup_{n=1}^{\infty} A_n$ $(i \leq n)$. We can find 0 open, in X, $p_i \in 0$, $p_j \notin 0$ for $j \neq i$, $1 \leq j \leq n$. Now $\left(p_i, \frac{1}{n}\right) \in 0 \times \left(\frac{1}{n} - \varepsilon, \frac{1}{n} + \varepsilon\right)$ for every ε and $0 \times \left(\frac{1}{n} - \varepsilon, \frac{1}{n} + \varepsilon\right)$ contains no number of $\bigcup_{n=1}^{\infty} A_n$ for sufficiently small ε. i. e. $\left(p_i, \frac{1}{n}\right)$ isolated in $\bigcup_{n=1}^{\infty} A_n$.

CLAIM. $\overline{\bigcup_{n=1}^{\infty} A_n} \supseteq D \times \{0\}$. For $(p_i, 0) \in 0' = 0_1 \times 0_2$ open, pick j large enough so that $1/j \in 0_2$ and $i \leq j$. As a result $\left(p_i, \frac{1}{j}\right) \in 0'$, $\left(p_i, \frac{1}{j}\right) \in \bigcup_{n=1}^{\infty} A_n$. Therefore $\overline{\bigcup_{n=1}^{\infty} A_n} \supseteq D$, $\overline{D} = X$ and $\left(\bigcup_{n=1}^{\infty} A_n\right) \cup X$ is a compactification of N.

Define a space X to be *residue complete* if it has an α-point compactification for every $1 \leq \alpha \leq |\beta X \setminus X|$. The rational numbers in $[0,1]$ are not residue complete as they have no n-point compactification for finite n and no $\aleph_0$-point compactification. In fact a non-locally compact space is not residue complete as it has no n-point compactification for finite n. The space N is residue complete, its residues including the one-point compactification of the discrete space of cardinal c, $[0,1]$ and of course $\beta N \setminus N$. R = real numbers with the usual topology has no

n-point compactification for $n > 2$, n finite. It would be interesting to know which spaces are residue complete. In particular which discrete spaces are. The difficulty would be in exhibiting an α-point compactification of discrete X where $\alpha = 2^{|X|}$.

REFERENCES

[1] E. CECH, *On bicompact spaces*, Ann. of Math., **38** (1937), pp. 823-844.

[2] L. GILLMAN and M. JERISON, *Rings of Continuous Functions*, Princeton, 1960.

[3] F. HAUSDORFF, *Uber zwei satze von G. Fichtenholz und L. Kantorovich*, Studia Math., **6** (1936), pp. 18-19.

[4] M. H. Stone, *Aplications of the theory of Boolean rings to general topology*, Trans. Amer. Math. Society, **41** (1937), pp. 375-481.