ON A CLASS OF NONLINEAR DEGENERATE PARABOLIC EQUATIONS

by

Miguel A. Herrero

Dpto. de Ecuaciones Funcionales
Facultad de matematicas,
Universidad Complutense
Madrid – 3. España

In memory of J. Sebastião e Silva

0. INTRODUCTION.

In this paper we report on some recent work on nonlinear degenerate parabolic equations made in collaboration with J. L. Vazquez ([10], [11]). To start with, let us consider the following problem:

\[ (0.1) \quad u_t = (\phi (u_x))_x \quad \text{in} \quad Q = \{(x, t): x \in \mathbb{R}, t > 0\} \]

\[ (0.2) \quad u(x, 0) = u_0(x) \quad \text{in} \quad \mathbb{R}. \]

We will assume \( \phi \) to be a continuous, nondecreasing function such that \( \phi (0) = 0 \) and \( \lim_{|s| \to \infty} |\phi (s)| = \infty \). An important case appears when \( \phi (s) = |s|^{m-1} s \) with \( m > 0 \). Then equation (0.1) reads

\[ (0.3) \quad u_t = (|u_x|^{m-1} u_x)_x \quad \text{in} \quad Q \]

Of course, if \( m = 1 \) (0.3) is the well-known linear heat equation which will not be considered here, so that we will assume \( m \neq 1 \). There are several physical problems where equations (0.1), (0.3) are relevant: see [4], [14]. On the other hand, if we write \( w = u_x \) and differentiate in (0.1) with respect to \( x \) we obtain

\[ (0.4) \quad w_t = (\phi (w))_{xx} \quad \text{in} \quad Q. \]

Received June 6, 1982. Revised July 29, 1982.
When $\phi(s) = |s|^{m-1}s$ with $m > 0$ this is known as the porous media equation. Much is known concerning the solutions of (0.4) with non-negative initial data $w(x, 0) = w_0(x)$, see [17]. Our results give information for the case where $w_0(x)$ is allowed to change sign, where most of the techniques developed to study the case of constant sign no longer apply.

A common feature of equations (0.1), (0.4) is the fact that they are formally parabolic if $u_\lambda \neq 0$ (resp. $u \neq 0$) whereas they degenerate if $u_\lambda = 0$ (resp. $u = 0$). Degeneracy is known to have a deep influence on the properties of solutions. For instance, if we consider equation (0.4) with $\phi(s) = |s|^{m-1}s$ and $m > 1$, it is known that if $w(x, 0)$ is nonnegative and compactly supported then any solution of the corresponding Cauchy problem in $Q$ must also be nonnegative and have compact support in $\mathbb{R}$ for each $t > 0$ and the transition from the region where $u > 0$ to the one where $u = 0$ is in general not smooth, so that it is necessary to deal with generalized solutions rather than with classical solutions [16], [3].

Here we try to give some insight on the corresponding situation for (0.1), (0.2). A first question to be considered is the following:

- In what a sense solutions to (0.1), (0.2) can be constructed? What can be said about classical solvability?

Next assume that $u_0(x)$ is compactly supported and define:

\begin{equation}
(0.5) \quad \zeta_(t) = \inf \{x \in \mathbb{R}: u(x, t) \neq 0\}, \quad \zeta_2(t) = \sup \{x \in \mathbb{R}: u(x, t) \neq 0\}.
\end{equation}

Of course $\zeta_1(t)$, $\zeta_2(t)$ may be equal to $+\infty$ (this is precisely what happens if $m = 1$). When they are finite for every $t > 0$, we call them interface curves or free boundaries. A second question we shall consider here is:

- What conditions on $\phi$ ensure the existence of interface curves for (0.1), (0.2)? If these curves exist, what can be said about their regularity as well as about the laws governing its growth?

When $\phi(s) = |s|^{m-1}s$ with $m > 0$ the type of behaviour we may expect is given by the following family of explicit self-similar solutions which is due to Barenblatt [6]:

\begin{equation}
(0.6) \quad \overline{u}(x, t) = \overline{u}_{x+}(x, t) = (t + \tau)^{-\frac{1}{2m}} \left[ a - \frac{m - 1}{m + 1} \left( \frac{1}{2m} \right)^\frac{1}{m} |x|^{\frac{m+1}{m}} \right] \cdot (t + \tau)^{\frac{m+1}{2m^2}} + \frac{m}{m - 1}
\end{equation}
where \( a > 0 \) and \( \tau \geq 0 \) are arbitrary and \( [s]_+ = \max \{ s, 0 \} \). We see that if \( m > 1 \) there are interfaces given by \( u_i(t) = \frac{c_i}{4} k (t + \tau)^{1/2m} \) with \( i = 1, 2 \), and
\[
K = [a (2m)^{1/m} \left( \frac{m + 1}{m - 1} \right)^{m/m + 1}, \text{whereas for } m < 1 \text{ solutions are everywhere positive. Concerning regularity, we see that } u \text{ is not smooth at points where } u_x = 0; \text{ in particular this is the case for the interfaces when } m > 1.
\]

Our plan is as follows: In Section 1 we recall briefly existence and uniqueness of generalized solutions, as it follows from standard results. In Section 2 we give stronger regularity statements for such solutions, derived under (rather weak) suitable hypotheses on \( \phi, u_0 \). Finally propagation properties, including a description of the behaviour of interfaces when they appear, are collected in Section 3. For complete proofs, the reader is referred to [10] and [11].

1. EXISTENCE AND UNIQUENESS.

Let \( B \) an operator in a Banach space \( X \). We say that a function \( u \in C ([0, T], X) \) with \( 0 < T \leq \infty \) is a strong solution of the problem
\[
\frac{du}{dt} (t) + Bu(t) = 0
\]
\[
u(0) = u_0
\]
where \( u_0 \in X \), if \( u \) is absolutely continuous in \( (0, T) \), \( u(0) = u_0 \) and for a.e. \( t > 0, u(t) \in D(B) \) (the domain of \( B \)) and
\[
\frac{du}{dt} = Bu; \text{ see[8]. We want to show that (0.1), (0.2) has a unique strong solution in } X = L^2(\mathbb{R}) \text{ (in fact strong solutions can be defined in } L^p(\mathbb{R}) \text{ for any } p \text{ such that } 1 \leq p < \infty). \text{ For this purpose, let us define the functional:}
\]
\[
J(u) = \begin{cases} 
\int j(u, \ldots) \, dx & \text{if } u \in L^2(\mathbb{R}), j(u, \ldots) \in L^1(\mathbb{R}) \\
+ \infty & \text{otherwise}
\end{cases}
\]
where \( j(s) = \int_0^s \phi(\xi) \, d\xi \). Then \( J \) is a convex, lower semicontinuous and proper (i.e. \( J(u) \geq -\infty \) for every \( u \in L^2(\mathbb{R}), J \equiv +\infty \) functional in \( L^2(\mathbb{R}), \))
and its subdifferential (generalized gradient) $B = \partial J$ is a maximal monotone operator in $L^2(\mathbb{R})$ given on the domain

$$D(B) = \{ u \in L^2(\mathbb{R}) : \text{there exists } w \in C(\mathbb{R}) \text{ such that}$$

$$w = \phi(u_x) \text{ a.e. and } w_x \in L^2(\mathbb{R})\}$$

by

$$Bu = -w_x$$

The above results are obtained by an easy modification of the work of Attouch and Damlamian [5]. Then the existence and uniqueness of strong solutions for (0.1), (0.2) follows from the results of Brezis [7].

2. REGULARITY.

Let us write $u(x, t)$ to mean the strong solution of (0.1), (0.2) with $u_0 \in L^2(\mathbb{R})$. Our first result is the following: for $\tau > 0$ we write $Q_\tau = \{(x, t) : x \in \mathbb{R}, t \geq \tau\}$. Then:

**Theorem 2.1.** $u(x, t)$ is continuous and bounded in $Q_\tau$ for each $\tau > 0$. In fact we have:

$$|u(x, t) - u(\overline{x}, \overline{t})| \leq C_1 |x - \overline{x}| + C_2 |t - \overline{t}|^{2/3}$$

for every $x, \overline{x} \in \mathbb{R}$ and $t, \overline{t} \geq \tau$. $C_1, C_2$ are positive constants depending on $\phi$, $\tau$ and $\|u_0\|_2$.

The proof consists of two steps; first one shows that for each $t > 0$ $u$ and $u_x$ are bounded, a regularizing effect. (see [2], [9]). The result follows from this and the estimate $\|u_t\|_2 \leq \|u_0\|_2 \tau^{-1}$ (7).

Assume now that in addition $\phi \in C^2(\mathbb{R} - \{0\})$ and $\phi'(s) > 0$ if $s \neq 0$ (for instance, this is the case for equation (0.3)). Then we have

**Theorem 2.2.** Let $\phi$ be as above. Then $u_x(x, t)$ is continuous and bounded in $Q_\tau$ for every $\tau > 0$. Moreover, $u(x, t)$ is a classical solution of (0.1) in $\Omega_0 = \{(x, t) \in Q : u_x(x, t) \neq 0\}$. 
The proof is obtained by setting a sequence of approximating problems $u^n_i = (\phi_n(u^n_i))$ in $Q$, $u^n(x, 0) = u_0$ which are non degenerate and whose solutions are classical and converge to $u(x, t)$ in the sense that for fixed $t > 0$, $u^n(x, t) \rightarrow u(x, t)$, $u^n_i(x, t) \rightarrow u_i(x, t)$ as functions of $x$, uniformly on compacts. For this, the concept of convergence of functionals in the sense of Mosco ([15]) plays a central role. Continuity in time for $u_i(x, t)$ follows by adapting a device of Kruzhkov [13] which gives Hölder continuity in $t$ from Hölder continuity in $x$; this is applied to the $u^n_i(x, t)$ in a family of sets which is shown to be independent of $n$, so that the conclusion follows by letting $n \rightarrow \infty$. Classical solvability in $\Omega_0$ follows then by standard Schauder-type estimates.

We remark that for the analogous to (0.3) with $N \geq 1$, $m > 1$ the continuity of the gradient has been shown independently in [1].

In case $\phi(s) = |s|^{m-1} s$, $0 < m < \infty$ (i.e., for equation (0.3)) the homogeneity of the operator can be used to obtain additional results.

**Theorem 2.3.** Let $u(x, t)$ be the strong solution of (0.3), (0.2) with $u_0 \in L^2(\mathbb{R})$. Then for each $\tau > 0$, $u$ is uniformly Lipschitz continuous in $x$, $t$ and $u_i(\cdot, t)$ is uniformly Hölder continuous in $x$ with exponent $\alpha = \min \left(1, \frac{1}{m}\right)$.

We remark that exponent $\alpha$ is best possible as can be seen by direct computation on (0.5).

3. PROPAGATION PROPERTIES.

Let us start with problem (0.3), (0.2). We note that equation (0.3) can be written as a conservation law as follows:

$$u_i + (u \cdot V)_x = 0$$

with

$$V = -|u_x|^{m-1} u_x u^{-1} \quad \text{if} \quad u \neq 0$$

We can interpret $V$ as the velocity with which disturbances with respect to the level $u = 0$ propagate. We first show that if $m > 1$ and $u_0 \in L^p(\mathbb{R})$, $1 \leq p < \infty$, $u_0 \geq 0$ the velocity is bounded in $Q$ for every $\tau > 0$:
Theorem 3.1. Let \( u_0, m \) be as above. Then there exists a constant \( C = C(m, p) > 0 \) such that for \( t > 0 \):

\[
(3.1) \quad |u_x|^m \leq Cu_0 \| u_0 \|_p^\alpha t^{-\beta}
\]

with \( \alpha = \frac{pm + m - 1}{p(m + 1) + m - 1} \), \( \beta = \frac{pm + m - 1}{p(m + 1) + m - 1} \) if \( p < \infty \) and \( \alpha = \frac{m - 1}{m + 1} \), \( \beta = \frac{m}{m + 1} \) if \( p = \infty \).

The proof starts by obtaining the bound \( |u_x|^m \leq K \| u \| \) with \( K = K(m, p, \| u_0 \|_p, t) \) by means of a classical device of Bernstein for bounding the gradient of elliptic and parabolic equations, as adapted by Aronson [3] for the porous media equation. The main difficulty here is technical, for the computations involved require three derivatives whereas no approximation by global smooth solutions is available. Once this has been established we take advantage of the homogeneity of (0.3) by rescaling with respect to the time and initial condition. This gives (3.1).

When \( u_0(x) \) has bounded velocity (i.e., when \( u_0^{(m-1)/m} \) is Lipschitz) the bound \( |u_x|^m \leq K \| u \| \) for some \( K > 0 \) has been obtained in [12].

Estimate (3.1) is crucial in deriving existence and regularity of interfaces, as well as estimates on their growth. In particular, if we recall the definition of \( \zeta_i(t) (i = 1, 2) \) given in (0.5), \( m \) and \( u_0 \) are as in Theorem 3.1 and \( u_0 \) is also compactly supported, we obtain:

Theorem 3.2. For \( i = 1, 2 \), \( (-1)^i \zeta_i(t) \) is continuous non decreasing in \([0, \infty[\), uniformly Lipschitz continuous away from zero and, if the support of \( u_0 \) is given by \([a_1, a_2]\) then:

\[
(3.2) \quad (-1)^i (\zeta_i(t) - a_i) \leq C_1 \| u_0 \|_p^\frac{p}{p - 1} \cdot t^\gamma
\]

with \( C_1 = C_1(m, p) > 0 \), \( \gamma = \frac{p}{p(m + 1) + m - 1} \) if \( p < \infty \), \( \gamma = \frac{1}{m + 1} \) if \( p = \infty \).

We remark that by the maximum principle \( u(x, t) \geq 0 \) if \( u_0(x) \geq 0 \) so that \( u \geq 0 \) in \((\zeta_1(t), \zeta_2(t))\). On the other hand the exponents in (3.1), (3.2) are given a priori by the homogeneity of the operator; thus they are the only
possible ones. The best exponent for large time behaviour of the \( \zeta_i \)'s in (3.2) corresponds to dependence respect to \( L^1 \)-norm, where we have:

\[
(-1)^k (\zeta_i(t) - a_i) \leq C_i \| u_0 \|_{L^m} \cdot t^{\frac{m-1}{2m}}
\]

(compare with (0.5)). In fact, using a device introduced by Vazquez [18] for the porous media equation, we can give a precise result on the asymptotic behaviour of the interfaces.

**Theorem 3.3.** Let \( u_0, a_i, a_j \) as before. Then:

\[
\lim_{t \to \infty} \frac{(-1)^k \zeta_i(t)}{t^{1/2m}} = b \| u_0 \|_{L^m} \cdot t^{\frac{m-1}{2m}}
\]

where \( b = b(m) > 0 \).

This is precisely the behaviour of Barenblatt's solutions (see (0.5)).

When \( m < 1 \) the example given in (0.5) suggests that no interfaces exist. In fact we prove:

**Theorem 3.4.** Let \( m < 1, u_0 \geq 0 \) and \( u_0 \equiv 0 \). Then for each \( x \in \mathbb{R}, t > 0 \) we have \( u(x, t) > 0 \).

If we consider the general problem (0.1), (0.2) we obtain a similar situation from a qualitative viewpoint, in the sense that there is a condition which is necessary and sufficient for the existence of interfaces, namely:

\[
\int_{-1}^1 \frac{ds}{\phi^{-1}(s)} < + \infty
\]

(note that for \( \phi(s) = |s|^{m-1} \) this means \( m > 1 \)). Estimates on the behaviour of \( \zeta_i(t) \), depending on \( \phi \), are also available in this case.

**REFERENCES**


