ON THE DUNFORD-PETTIS PROPERTY

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A Banach space $E$ is said to have the Dunford-Pettis property (DPP in short) if every weakly compact operator on $E$ transforms weakly convergent sequences into norm convergent ones. This last class of operators are called, by obvious reasons, Dunford-Pettis operators. The DPP was introduced by Grothendieck in his remarkable paper [7] and has been intensively studied (see [4]). The long standing open question of whether the Banach space of all the continuous $E$-valued functions on the compact Hausdorff space $K$ has the DPP if $E$ has, was answered in the negative by Talagrand, who built in [10] a Banach space $H$ with the DPP and a weakly compact operator from $C([0,1], H)$ into $c_0$ which is not a Dunford-Pettis operator. In [1] a class of operators on $C(K, E)$ spaces, more general than the Dunford-Pettis operators are introduced, namely the almost Dunford-Pettis operators, and it turns out that the weakly compact, non Dunford-Pettis operator exhibited by Talagrand in [10], is in fact almost Dunford-Pettis. Hence, the following natural question arises (see [1]): which are the Banach spaces $E$ such that, regardless the compact Hausdorff space $K$, every weakly compact operator on $C(K, E)$ is almost Dunford-Pettis? In this note we prove that such spaces are precisely those with the DPP, giving so a new operator characterization of this property.

Before proceeding any further, let us fix some additional terminology: Recall that every operator (= linear bounded operator) $T$ from

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$C(K, E)$ into another Banach space $F$ has a representing measure \( m \), i.e., a finitely additive measure of bounded semivariation defined on the Borel $\sigma$-field $\text{Bo}(K)$ of $K$, with values in $L(E, F^{**})$, in such a way that

$$T(f) = \int f \, dm \quad \text{for each} \quad f \in C(K, E)$$

(see f.i., [5] or [6]). In particular, the topological dual of $C(K, E)$ can be identified with the space $\text{rcabv}(\text{Bo}(K), E^*)$ of all the $E^*$-valued, regular, countably additive Borel measures on $K$ of bounded variation, endowed with the variation norm. If $m: \text{Bo}(K) \rightarrow L(E, F)$ is a finitely additive measure, we shall denote by $|m|$ its semi-variation ([6], pag. 51). $|m|$ is said to be continuous at $\emptyset$ if it has a control measure, i.e., a positive Radon measure $\lambda$ on $K$ such that $\lim_{\lambda(A) \to 0} |m|(A) = 0$. For the rest of notation and terminology used and not defined along the paper, we refer to [5] or [6].

The following well known result is stated for reference purposes (see [6], sect. 13, Th. 5):

**Theorem 1.** Let $\lambda$ be a positive Radon measure on $K$. If $m \in \text{rcabv}(\text{Bo}(K), E^*)$ is $\lambda$-continuous, there exists a function $g$ from $K$ into $E^*$ such that

a) $\langle x, g \rangle$ is a $\lambda$-integrable function for every $x \in E$.

b) For every $x \in E$ and $A \in \text{Bo}(K)$,

$$\langle x, m(A) \rangle = \int_A \langle x, g \rangle \, d\lambda .$$

c) The function $t \mapsto |g|(t) = \|g(t)\|$ is $\lambda$-integrable and

$$|m|(A) = \int_A |g| \, d\lambda \quad \text{for} \quad A \in \text{Bo}(K) .$$

d) If $\rho$ is a lifting of $\mathcal{L}^\infty(\lambda)$, we can choose $g$ uniquely $\lambda$-almost everywhere such that $\rho[g] = g$. 

Recall that a subset $B$ of a Banach space is said to be weakly conditionally compact if every sequence in $B$ has a weak Cauchy subsequence. Rosenthal's $\ell^1$ theorem ([8]) can be stated by saying that every bounded subset of a Banach space is either weakly conditionally compact or it contains an $\ell^1$-sequence, i.e., a sequence equivalent to the usual basis of $\ell^1$. 
The next result can be proved using the same argument as in the proof of prop. 3.1 (ii) of [3]:

**Lemma 2.** Let $B$ be a weakly conditionally compact subset of $\text{rcabv} (\text{Bo} (K), E)$. Then

$$|B| = \{|m|: m \in M\} \subset \text{rcabv} (\text{Bo} (K))$$

is uniformly countably additive. In consequence, there exists a control measure for $|B|$. □

Now, if $K$ is a compact Hausdorff space and $\phi \in C(K)$, for every Banach space $E$ the linear map $S_\phi: C(K, E) \to C(K, E)$ defined by $S_\phi (f) = \phi f$ is continuous. For a measure $m \in \text{rcabv} (\text{Bo} (K), E^*) = C(K, E)^*$, let us write $S^\phi_\phi (m) = \phi m$. With this notation, we have the following result, which extends a previous one of Bourgain for spaces of Bochner integrable functions ([2], prop. 10):

**Lemma 3.** If $B$ is a weakly conditionally compact subset of $\text{rcabv} (\text{Bo} (K), E^*)$ and $D$ is a bounded subset of $C(K)$, then

$$D \cdot B = \{ \phi m: \phi \in D, m \in B\}$$

is also weakly conditionally compact in $\text{rcabv} (\text{Bo} (K), E^*)$.

**Proof:** If the statement were untrue, by Rosenthal’s $\ell^1$ theorem there would be sequences $(m_n)$ in $B$ and $(\phi_n)$ in $D$ such that $(\phi_n m_n)$ is an $\ell^1$-sequence, i.e., for some $r > 0$,

$$\left\| \sum a_n \phi_n m_n \right\| \geq r \sum |a_n|,$$

for all finitely non zero real sequences $(a_n)$.

By lemma 2, there exists a positive Radon measure $\lambda$ on $K$ such that every member of $B$ is absolutely continuous with respect to $\lambda$. Let $\rho$ be a lifting of $\mathcal{L}^\infty (\lambda)$ and $g_n$ the function corresponding to $m_n$ by theorem 1.
Let us note that
\[
\langle f, \phi_n m_n \rangle = \langle \phi_n f, m_n \rangle
\]
\[
= \int \langle \phi_n f, g_n \rangle \, d\lambda
\]
\[
= \int \langle f, \phi_n g_n \rangle \, d\lambda
\]
\[
= \int \langle f, \rho(\phi_n) g_n \rangle \, d\lambda
\]
for every \( n \in \mathbb{N} \) and \( f \in C(K, E) \). On the other hand, for every finitely non zero sequence \((a_n)\) of reals, we have
\[
\rho \left[ \sum a_n \rho(\phi_n) g_n \right] = \sum a_n \rho(\phi_n) g_n .
\]
Hence, \( \sum a_n \rho(\phi_n) g_n \) is the function corresponding to \( \sum a_n \phi_n m_n \) by theorem 1, and so
\[
\left\| \sum a_n \phi_n m_n \right\| = \int \left| \sum a_n \rho(\phi_n) g_n \right| \, d\lambda .
\]
Now the proof goes as in proposition 10 of [2]: Let \((r_n)\) be the sequence of Rademacher functions on \([0, 1]\). By the "contraction principle" ([9], Exp. VII), we have
\[
M \int \left\| \sum a_n r_n(t) m_n \right\| \, dt \geq
\]
\[
\geq \int \int \left\| \sum a_n r_n(t) \rho(\phi_n)(s) g_n(s) \right\| \, d\lambda(s) \, dt =
\]
\[
= \int \left\| \sum a_n r_n(t) \phi_n m_n \right\| \, dt \geq r \sum |a_n| ,
\]
where \( M \) is a bound for \( \{ \| \phi_n \| : n \in \mathbb{N} \} \). Proposition 2 of [2] applies to produce an \( \ell^1 \)-subsequence of \((m_n)\). This contradiction proves the lemma.

**Definition 4.** ([1], def. 1.8) An operator \( T \) from \( C(K, E) \) into \( F \) whose representing measure has semivariation continuous at \( \emptyset \) is called almost Dunford-Pettis if for every weakly null sequence \((x_n)\) in \( E \) and every bounded sequence \((\phi_n)\) in \( C(K) \), we have \( \lim_{n \to \infty} T(\phi_n x_n) = 0 \).

Now we can prove the announced result:
Theorem 5. Let $E$ be a Banach space. The following assertions are equivalent:

a) For any compact Hausdorff space $K$, every weakly compact operator on $C(K, E)$ is almost Dunford-Pettis.

b) Every weakly compact operator on $C([0,1], E)$ is almost Dunford-Pettis.

c) Every weakly compact operator from $C([0,1], E)$ into $c_0$ is almost Dunford-Pettis.

d) For every weakly null sequence $(x_n)$ in $E$ and every weakly null sequence $(m_n)$ in $rcabv(Bo ([0,1]), E^*)$, we have

$$
\lim_{n \to \infty} \int |\langle x_n, g_n \rangle| \, d\lambda = 0 ,
$$

where $\lambda$ is a common control measure for the sequence $(m_n)$ and $g_n$ is the function corresponding to $m_n$ and $\lambda$ by Theorem 1.

e) $E$ has the Dunford-Pettis property.

Proof: a) $\Rightarrow$ b) and b) $\Rightarrow$ c) are clear.

c) $\Rightarrow$ d) Let $(x_n)$ and $(m_n)$ be weakly null sequences as in d). The map $T: C([0,1], E) \to c_0$, defined by

$$
T(f) = \left( \langle f, m_n \rangle \right)_{n=1}^{\infty} ,
$$
is obviously linear continuous, and $T^*(e_n) = m_n$, where $(e_n)$ denotes the usual unit basis of $\ell^1 = c_0^*$. Then $T^*$ is weakly compact, and so is $T$ by Gantmacher's theorem. Hence, by hypothesis, $T$ is almost Dunford-Pettis. Let $\phi_n$ be a continuous scalar function on $[0,1]$ such that $\|\phi_n\| \leq 1$ and

$$
\int |\langle x_n, g_n \rangle| \, d\lambda \leq \int \phi_n \langle x_n, g_n \rangle \, d\lambda + \frac{1}{n} .
$$

Then

$$
\int |\langle x_n, g_n \rangle| \, d\lambda \leq \int \phi_n \langle x_n, g_n \rangle \, d\lambda + \frac{1}{n} \leq |T(x_n \phi_n)| + \frac{1}{n} ,
$$

which tends to zero when $n$ tends to $\infty$. 

Take weakly null sequences \((x_n)\) and \((x_n^*)\) in \(E\) and \(E^*\), respectively. Let \(\lambda\) be the Lebesgue measure on \([0, 1]\). The linear map \(E^* \ni x^* \mapsto x^*\lambda \in \text{rcabv}(B_0([0, 1]), E^*)\) is obviously continuous, and so \((x_n^*\lambda)\) is weakly null. The function corresponding to \(x_n^*\lambda\) and \(\lambda\) by theorem 1 is clearly the constant function which takes the value \(x_n^*\) at every point. Then, by hypothesis,

\[
\lim_{n \to \infty} \int |\langle x_n, x_n^* \rangle| \ d\lambda = \lim_{n \to \infty} |\langle x_n, x_n^* \rangle| = 0,
\]

and so \(E\) has the DPP (see [4], Th. 1(c), for example).

Let us consider a weakly null sequence \((x_n)\) in \(E\), a bounded sequence \((\phi_n)\) in \(C(K)\) and a weakly compact operator \(T\) from \(C(K, E)\) into another Banach space \(F\). If \((T(\phi_n x_n))\) does not converge to 0, passing to a subsequence if necessary, we can suppose that there exists an \(r > 0\) such that \(|T(\phi_n x_n)| > r\) for all \(n\). Choose an \(y_n^* \in F^*\) verifying \(|y_n^*| \leq 1\) and

\[
\langle T(\phi_n x_n), y_n^* \rangle > r, \quad \text{for every} \quad n.
\]

If \(m_n = T^*(y_n^*)\), the set \(\{m_n : n \in \mathbb{N}\}\) is then weakly relatively compact. So, by lemma 3, \(\{\phi_nm_n : n \in \mathbb{N}\}\) is weakly conditionally compact. Therefore, there exists a weakly Cauchy subsequence \((\phi_{n_k} m_{n_k})\). Now, for every Borel set \(A\) in \([0, 1]\), the linear map \(\text{rcabv}(B_0([0, 1]), E^*) \ni m \mapsto m(A)\) is clearly continuous; consequently, \(x_{n_k}^* = (\phi_{n_k} m_{n_k})([0, 1])\) forms a weakly Cauchy sequence in \(E^*\). If \(\lambda\) is a common control measure for \((m_n)\) and \(g_n\) is the function corresponding to \(m_n\) and \(\lambda\) by theorem 1, then

\[
\langle T(\phi_{n_k} x_{n_k}), y_{n_k}^* \rangle = \int \langle x_{n_k} \phi_{n_k}, g_{n_k} \rangle \ d\lambda = \int \langle x_{n_k}, \phi_{n_k} g_{n_k} \rangle \ d\lambda = \langle x_{n_k}, x_{n_k}^* \rangle.
\]

But \(E\) having the DPP implies that \(\lim_{k \to \infty} \langle x_{n_k}, x_{n_k}^* \rangle = 0\) (see, for instance, [4], Th. 1(f)), contradicting \((\#)\).

Finally, it should be mentioned that reasoning as in Th. 3.2 of [1], one can prove:
Proposition 6. For a Banach space $E$, the following conditions are equivalent:

a) For some non dispersed compact Hausdorff space $K$, every weakly compact operator on $C(K, E)$ is almost Dunford-Pettis.

b) Every weakly compact operator on $C([0, 1], E)$ is almost Dunford-Pettis.

c) $E$ has the Dunford-Pettis property. 

REFERENCES


[10] Talagrand, M. – *La propriété de Dunford-Pettis dans* $C(K, E)$

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